# Operations Research Techniques and Algorithms (620-361)

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# Today's Lecture

The function $f$ is *unimodal* on $[a, b]$ if it has only one local minimum - note, this local minimum is thus also a global minimum.

Until further notice, we shall assume that $f$ is unimodal.

Now assume that we have two $f$-calculations at points $p$ and $q$. That is, we know $f(p)$ and $f(q)$. A little thought shows us that

- If $f(p) \leq f(q)$, then $x_{min} \in [a, q]$.
- If $f(p) \geq f(q)$, then $x_{min} \in [p, b]$.

Now suppose that we want to find $x_{min}$ to within a specified tolerance $\epsilon$. That is we want to reduce the size of the interval in which we know the minimum lies to width $2\epsilon$.

How do we choose $p$, $q$ and all the subsequent points at which we make an $f$-calculation so that we get to this stage with as few $f$-calculations as possible?

## The Fibonacci Search

Suppose that we are allowed $n \geq 2$ $f$-calculations, and that we want to reduce the length of the interval in which the minimum occurs from $[a, b]$ to the smallest length possible.

Let's call this smallest length $\alpha$ and, for $k \leq n$, define $F_k(\alpha)$ to be the maximum length of an interval that can be reduced to $\alpha$ in $k$ $f$-calculations.

Thus $F_n(\alpha) = b - a$. How should we proceed?

We make two $f$ calculations at points $p < q$. Then, by the observation above, the new interval will be either $[a, q]$ or $[p, b]$. We can't tell which will be the case, so we should choose $p$ and $q$ such that $q - a = b - p$.

For the purposes of this explanation, let's assume (w.l.o.g.) that $f(p) \leq f(q)$ and so $x_{min} \in [a, q]$.

We still have $n-2$ $f$-calculations to use to reduce the interval. Consider first the case $n>2$.

We already know the value of $f$ at point $p \in [a, q]$, and it makes sense to use this. Thus, we let the length of $[a, q]$ (and $[p, b]$) be denoted $F_{n-1}(\alpha)$.

In order to reduce the interval again, we need to evaluate $f$ at a new point $r$, then compare $f(r)$ and $f(p)$. By the same argument as we used above, we should choose $r$ such that $p-a=q-r$.

Both of the intervals $[a, p]$ and $[r, q]$ are of length $F_{n-2}(\alpha)$.

We can now write the relation

$$b - a = (b - p) + (p - a)$$

as

$$F_n(\alpha) = F_{n-1}(\alpha) + F_{n-2}(\alpha). \tag{1}$$

If $n = 2$, then the situation is a bit different; we have no more $f$ calculations to use (except for the "basic" first two, without which we can't do anything). Thus, we want to make $q - a$ and $p - b$ as small as possible, which we can do by making them both equal to $(b - a)/2$. Therefore we take $q = p = a + (b - a)/2$.

In practice, we can't do this exactly because we need $p < q$, but we can take $p$ and $q$ arbitrarily close to this point.

This gives us

$$F_2(\alpha) = 2\alpha \quad \text{(approximately)}. \tag{2}$$

Also, we can't reduce the length of the interval at all with zero or one $f$-calculation, so it is reasonable to let

$$F_0(\alpha) = \alpha \tag{3}$$

and

$$F_1(\alpha) = \alpha. \tag{4}$$

Equations (1) define the *Fibonacci sequence*.

The initial conditions are given by (3) and (4). Thus we see that $F_n(\alpha) = F_n\alpha$ where $F_n$ is given by the Fibonacci sequence: $F_0 = 1$, $F_1 = 1$, $F_2 = 2$, $F_3 = 3$, $F_4 = 5$, $F_5 = 8$, $F_6 = 13$, $F_7 = 21$, $F_8 = 34$, $F_9 = 55$, $F_{10} = 89 \ldots$.

## Fibonacci Search Algorithm

To minimise a unimodal function $f$ over $[a, b]$ to within tolerance $\epsilon$.

1. Find the smallest value of $n$ such that $(b-a)/F_n < 2\epsilon$.
2. Set

$$\begin{aligned} k &= n \\ p &= b - \frac{F_{k-1}}{F_k}(b-a) \\ q &= a + \frac{F_{k-1}}{F_k}(b-a) \end{aligned}$$

   Calculate $f(p)$ and $f(q)$.

3. Set $k = k - 1$. If $f(p) \leq f(q)$, then set

$$\begin{aligned} b &= q \\ q &= p \\ p &= b - \frac{F_{k-1}}{F_k}(b - a) \end{aligned}$$

Calculate $f(p)$. If $f(p) > f(q)$, then set

$$\begin{aligned} a &= p \\ p &= q \\ q &= a + \frac{F_{k-1}}{F_k}(b - a) \end{aligned}$$

Calculate $f(q)$. Repeat until $k = 3$.

4. If $f(p) \leq f(q)$, then set

$$\begin{aligned} b &= q \\ q &= p \\ p &= b - 2\epsilon \end{aligned}$$

Calculate $f(p)$.
If $f(p) > f(q)$, then set

$$\begin{aligned} a &= p \\ p &= q \\ q &= a + 2\epsilon \end{aligned}$$

Calculate $f(q)$.

5. If $f(p) \leq f(q)$, then $b = q$.
   If $f(p) > f(q)$, then $a = p$.
   The final interval is $[a, b]$. This interval has length either $2\epsilon$ or $\alpha < 2\epsilon$.