Operations Research Techniques and Algorithms (620-361)

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Today’s Lecture

Unimodal 1-D unconstrained optimisation

Fibonacci search
The function $f$ is *unimodal* on $[a, b]$ if it has only one local minimum - note, this local minimum is thus also a global minimum.

Until further notice, we shall assume that $f$ is unimodal.
Now assume that we have two $f$-calculations at points $p$ and $q$. That is, we know $f(p)$ and $f(q)$. A little thought shows us that

- If $f(p) \leq f(q)$, then $x_{min} \in [a, q]$.
- If $f(p) \geq f(q)$, then $x_{min} \in [p, b]$.
Now suppose that we want to find $x_{min}$ to within a specified tolerance $\epsilon$. That is we want to reduce the size of the interval in which we know the minimum lies to width $2\epsilon$.

How do we choose $p$, $q$ and all the subsequent points at which we make an $f$-calculation so that we get to this stage with as few $f$-calculations as possible?
The Fibonacci Search

Suppose that we are allowed \( n \geq 2 \) \( f \)-calculations, and that we want to reduce the length of the interval in which the minimum occurs from \([a, b]\) to the smallest length possible.

Let’s call this smallest length \( \alpha \) and, for \( k \leq n \), define \( F_k(\alpha) \) to be the maximum length of an interval that can be reduced to \( \alpha \) in \( k \) \( f \)-calculations.

Thus \( F_n(\alpha) = b - a \). How should we proceed?
We make two \( f \) calculations at points \( p < q \). Then, by the observation above, the new interval will be either \([a, q]\) or \([p, b]\). We can’t tell which will be the case, so we should choose \( p \) and \( q \) such that \( q - a = b - p \).

For the purposes of this explanation, let’s assume (w.l.o.g.) that \( f(p) \leq f(q) \) and so \( x_{\text{min}} \in [a, q] \).
We still have $n - 2$ $f$-calculations to use to reduce the interval. Consider first the case $n > 2$.

We already know the value of $f$ at point $p \in [a, q]$, and it makes sense to use this. Thus, we let the length of $[a, q]$ (and $[p, b]$) be denoted $F_{n-1}(\alpha)$.

In order to reduce the interval again, we need to evaluate $f$ at a new point $r$, then compare $f(r)$ and $f(p)$. By the same argument as we used above, we should choose $r$ such that $p - a = q - r$.

Both of the intervals $[a, p]$ and $[r, q]$ are of length $F_{n-2}(\alpha)$. 
We can now write the relation

\[ b - a = (b - p) + (p - a) \]

as

\[ F_n(\alpha) = F_{n-1}(\alpha) + F_{n-2}(\alpha). \] (1)
If \( n = 2 \), then the situation is a bit different; we have no more \( f \) calculations to use (except for the ”basic” first two, without which we can’t do anything). Thus, we want to make \( q - a \) and \( p - b \) as small as possible, which we can do by making them both equal to \((b - a)/2\). Therefore we take \( q = p = a + (b - a)/2 \).

In practice, we can’t do this exactly because we need \( p < q \), but we can take \( p \) and \( q \) arbitrarily close to this point.

This gives us

\[
F_2(\alpha) = 2\alpha \quad \text{(approximately).} \tag{2}
\]
Also, we can’t reduce the length of the interval at all with zero or one $f$-calculation, so it is reasonable to let

$$F_0(\alpha) = \alpha$$  \hspace{1cm} (3)

and

$$F_1(\alpha) = \alpha.$$  \hspace{1cm} (4)
Equations (1) define the *Fibonacci sequence*. 

The initial conditions are given by (3) and (4). Thus we see that $F_n(\alpha) = F_n \alpha$ where $F_n$ is given by the Fibonacci sequence: $F_0 = 1$, $F_1 = 1$, $F_2 = 2$, $F_3 = 3$, $F_4 = 5$, $F_5 = 8$, $F_6 = 13$, $F_7 = 21$, $F_8 = 34$, $F_9 = 55$, $F_{10} = 89$...
Fibonacci Search Algorithm

To minimise a unimodal function \( f \) over \([a, b]\) to within tolerance \( \epsilon \).

1. Find the smallest value of \( n \) such that \( (b - a)/F_n < 2\epsilon \).
2. Set

\[
\begin{align*}
k &= n \\
p &= b - \frac{F_{k-1}}{F_k}(b - a) \\
q &= a + \frac{F_{k-1}}{F_k}(b - a)
\end{align*}
\]

Calculate \( f(p) \) and \( f(q) \).
3. Set \( k = k - 1 \). If \( f(p) \leq f(q) \), then set

\[
\begin{align*}
    b &= q \\
    q &= p \\
    p &= b - \frac{F_{k-1}}{F_k}(b - a)
\end{align*}
\]

Calculate \( f(p) \). If \( f(p) > f(q) \), then set

\[
\begin{align*}
    a &= p \\
    p &= q \\
    q &= a + \frac{F_{k-1}}{F_k}(b - a)
\end{align*}
\]

Calculate \( f(q) \). Repeat until \( k = 3 \).
4. If $f(p) \leq f(q)$, then set

\begin{align*}
    b &= q \\
    q &= p \\
    p &= b - 2\epsilon
\end{align*}

Calculate $f(p)$.

If $f(p) > f(q)$, then set

\begin{align*}
    a &= p \\
    p &= q \\
    q &= a + 2\epsilon
\end{align*}

Calculate $f(q)$. 
5. If \( f(p) \leq f(q) \), then \( b = q \).
   If \( f(p) > f(q) \), then \( a = p \).

The final interval is \([a, b]\). This interval has length either \(2\epsilon\) or \(\alpha < 2\epsilon\).