Operations Research Techniques and Algorithms (620-361)

Dr Yao-ban Chan

y.chan@ms.unimelb.edu.au
Telephone: 8344 9073
Office: Room 198, Richard Berry Building

Christina Burt
c.burt@ms.unimelb.edu.au
Telephone: 8344 1797
Office: 139 Barry St

Friday 18th April, 2008

Today's Lecture

Constrained optimisation

- Karush-Kuhn Tucker conditions

Karush-Kuhn-Tucker conditions for nonlinear programs

Now we shall focus on problems of the form NLP with both inequality and equality constraints. The optimality conditions conditions for such problems were developed first by Karush, and then jointly by Kuhn and Tucker.
Recall the problem of interest is the *nonlinear program*,

$$\begin{array}{ll} \min\limits_{x} & f(x) \\ \text{subject to} & g(x) \leq 0, \quad h(x) = 0, \end{array} \tag{NLP}$$

where the vector inequality $g(x) \leq 0 \in \Re^p$ means $g_i(x) \leq 0$ for each $i = 1, \ldots, p$, and the vector equality $h(x) = 0$ means $h_j(x) = 0$ for each $j = 1, \ldots, q$.

We introduce the concept of an *active* constraint at a feasible point x^*. These are the constraints that are satisfied with equality.

All the equality constraints $h_j(x^*) = 0$ are active, and each inequality constraint $g_i(x^*) \leq 0$ such that $i \in I(x^*) := \{i : g_i(x^*) = 0\}$ is also active.

$I(x^*)$ is *the set of active inequality constraint indices*.

Note that if x^* is a local minimum of the inequality constrained problem, then x^* must also be a local minimum for a version of the same problem which has all inactive constraints at x^* discarded. That is to say, *inactive constraints at x^* don't matter*, and can be ignored in the statement of optimality conditions.

In contrast, at a local minimum, active inequality constraints can be treated to a large extent as equalities. That is, if x^* is a local minimum of the inequality constrained problem, then x^* is also a local minimum for the equality constrained problem

$$
\begin{array}{ll}
\min\limits_{x} & f(x) \\
\text{subject to} & h(x) = 0 \\
\text{and} & g_i(x) = 0 \quad \text{for all} \quad i \in I(x^*).
\end{array}
$$

Therefore, treating this temporarily as an equality constrained problem, Theorem 4 suggests that if one of the constraint qualifications holds at x^*, then there should exist Lagrange multipliers $\eta_i^*, i = 1, \ldots, q$, and $\lambda_j^*, j \in I(x^*)$ such that

$$\nabla_x L(x^*, \lambda^*, \eta^*) = \nabla f(x^*) + \sum_{j \in I(x^*)} \lambda_j \nabla g_j(x^*) + \sum_{i=1}^{q} \eta_i \nabla h_i(x^*) = 0$$

If the Lagrange multipliers corresponding to inactive constraints are set to zero, we can write this as

$$\nabla_x L(x^*, \lambda^*, \eta^*) = \nabla f(x^*) + \sum_{j=1}^{p} \lambda_j \nabla g_j(x^*) + \sum_{i=1}^{q} \eta_i \nabla h_i(x^*) = 0$$

where we have set $\lambda_j^* = 0$ for $j \notin I(x^*)$. This motivates the following...

The Lagrangian function for (NLP) is

$$\begin{aligned} L(x, \lambda, \eta) &= f(x) + \sum_{i=1}^{p} \lambda_i g_i(x) + \sum_{j=1}^{q} \eta_j h_j(x) \\ &= f(x) + \langle \lambda, g(x) \rangle + \langle \eta, h(x) \rangle, \end{aligned}$$

where $\lambda \in \Re^p$ is the multiplier corresponding to $g(x)$ and $\eta \in \Re^q$ is the multiplier corresponding to $h(x)$. The vectors λ and η are generally called Lagrange or KKT multipliers.

Theorem 7: Let f, g and h be C^1 functions, and assume that one of the constraint qualifications (discussed below) on g and h holds at x^*. If x^* is a local minimum of (NLP), then there exist $\lambda^* \in \Re^p$ and $\eta^* \in \Re^q$ such that

KKTa. $0 = \nabla_x L(x^*, \lambda^*, \eta^*)$, that is

$$\begin{aligned} 0 &= \nabla f(x^*) + \sum_i \lambda_i^* \nabla g_i(x^*) + \sum_j \eta_j^* \nabla h_j(x^*) \\ &= \nabla f(x^*) + \nabla g(x^*)\lambda^* + \nabla h(x^*)\eta^*. \end{aligned}$$

KKTb.

0.1 $g(x^*) \leq 0$,
0.2 $\lambda^* \geq 0$, and
0.3 for each i, $\lambda_i^* g_i(x^*) = 0$.

KKTc. $h(x^*) = 0$.

If x^* is a local minimum, the KKT conditions are satisfied; however, the KKT conditions may be satisfied without x^* being a local minimum.

If the KKT conditions are satisfied at (x^*, λ^*, η^*), we say that (x^*, λ^*, η^*) is a *KKT point*. KKT points are basically stationary points of the non-linear program.

Let's investigate the KKT conditions by looking at a one-dimensional problem with a single inequality constraint,

$$\min_{x \in \Re} f(x) \quad \text{subject to} \quad x \geq 5. \tag{1}$$

Suppose x^* is a local minimum. If $x^* > 5$, then x^* is also a local minimum of the unconstrained problem, $\min f(x)$, hence $\nabla f(x^*) = 0$. Thus x^* is stationary for NLP, taking $\lambda^* = 0$.

If $x^* = 5$, then x^* need not be stationary for the unconstrained problem, but $f'(5) \geq 0$, because if $f'(5) < 0$ then $d = 1$ is a descent direction for f at $x^* = 5$, hence $f(5 + t) < f(5)$ for some small $t > 0$. This would contradict $x^* = 5$ being a local minimum of the constrained problem.

Let's compare the above observations with the KKT conditions. Let $g(x) = 5 - x$. We have no equality constraints $h(x)$ (that is $q = 0$). For $\lambda \in \Re$,

$$L(x, \lambda) \;=\; f(x) + \lambda(5 - x), \quad \nabla_x L(x, \lambda) \;=\; \nabla f(x) - \lambda.$$

KKTa says $\nabla f(x^*) = \lambda^*$; and KKTb says

1. $5 - x^* \leq 0$,
2. $\lambda^* \geq 0$, and
3. either $5 - x^* = 0$ or $\lambda^* = 0$.

The third condition, KKTc, is vacuous because there are no equality constraints.

If $x^* > 5$, KKTb(c) implies that $\lambda^* = 0$, hence $\nabla f(x^*) = 0$ from KKTa. If $x^* = 5$, λ is some nonnegative number and we deduce, from KKTa, that $\nabla f(x^*) \geq 0$.

Thus the KKT conditions are just the algebraic statement of our intuitive observations about the gradient of f at a solution point.

We have different "behaviour" depending upon whether the optimal solution is on a "boundary" of an inequality constraint set (constraint is active), or in the "interior" of this set (constraint is inactive).

Example. Find all KKT points of the 1-dimensional problem with a single inequality constraint, $\min_{x \in \Re} e^x \quad \text{subject to} \quad x \geq 5$.

Find all KKT points of the problem with the inequality reversed: $x \leq 5$.

Find all KKT points of the problem with an equality constraint: $x = 5$.