Operations Research Techniques and Algorithms (620-361)

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Today's Lecture

- Constrained optimisation
  - Constraint qualifications

Constraint qualifications

The two simplest constraint qualifications mimic the equality constrained case.

- The first is that g_i, for $i \in I(x^*)$, and h (or h_j, for each j,) are affine functions.
- The second is that the set of active gradients

  $$\{\nabla g_i(x^*) \,:\, i \in I(x^*)\} \cup \{\nabla h_j(x^*) :\ \text{all } j\} \tag{1}$$

  are linearly independent.

Theorem 7 holds if a point x^* satisfies one of the above constraint qualifications.

Theorem 7 can be used to identify candidate points for local minima. This is done by separately considering all possible combinations of constraints being active or inactive.

Example:

$$\begin{array}{llll} \text{min} & f(x) = & \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) & \\ \text{s.t.} & g_1(x) = & x_1 + x_2 + x_3 + 3 & \leq 0 \end{array}$$

A weaker condition than the linear independence constraint qualification (1) is the Mangasarian-Fromovitz constraint qualification.

This requires that the equality constraint gradients $\nabla h_j(x^*)$ are linearly independent, and there exists $d \in \Re^n$ such that

1. $\nabla h(x^*)^T d = 0$ and,
2. for $i \in I(x^*)$, $\nabla g_i(x^*)^T d < 0$.

The first condition implies that d is orthogonal to the space spanned by the gradients of the equality constraints $\nabla h_j(x^*)$ and the second condition requires that d is a "descent direction" for each of the inequality constraints.

Essentially, it is saying that none of the inequality constraints are implied by the equality constraints at the point x^*.

So Theorem 7 also holds if x^* satisfies the Mangasarian-Fromovitz constraint qualification.

Example. For the NLP

$$\begin{array}{lllll} \text{min} & x_1 & & & \\ \text{such that} & g_1(x) \overset{def}{=} (x_1-1)^2 & + & (x_2-1)^2 - 1 \le 0 \\ & g_2(x) \overset{def}{=} (x_1-1)^2 & + & (x_2+\frac{1}{2})^2 - 1 \le 0. \end{array}$$

sketch the feasible region, and explain, using the sketch, why $x^* = (1 - \frac{\sqrt{7}}{4}, \frac{1}{4})$ is a local (and global) minimum.

Show that the Mangasarian-Fromowitz condition holds at x^* (ie: find a suitable $d \in \Re^2$). Determine KKT multipliers for x^* and hence show that x^* is a KKT point, as predicted by Theorem 7.

In the following example, we shall see a case where the Mangasarian-Fromovitz constraint qualification does *not* hold at a local minimum, and observe that it is not possible to find KKT multipliers at this point, that is we show how Theorem 7 breaks down without the constraint qualification.

Consider the NLP

$$\begin{array}{ll} \min & x_1 + x_2 + x_3^2 \\ \text{such that} & g_1(x) \overset{def}{=} (x_1 - 1)^2 \quad + \quad (x_2 - 1)^2 - 1 \leq 0 \\ & g_2(x) \overset{def}{=} (x_1 - 1)^2 \quad + \quad (x_2 + 1)^2 - 1 \leq 0. \end{array}$$

It is not hard to see that the feasible set for this NLP is $\{(1,0,x_3) \ : \ x_3 \in \Re\}$ and thus the optimal solution must be $x^* = (1,0,0)$, which must also be a local minimum.

Note that there are no equality constraints, so to check the Mangasarian-Fromowitz constraint qualification, we seek $d \in \Re^3$ such that $\nabla g_i(x^*)^T d < 0$ for all i with $g_i(x^*) = 0$.

$$
\begin{aligned}
\nabla g_1(x) &= (2(x_1-1), 2(x_2-1), 0) \\
\nabla g_2(x) &= (2(x_1-1), 2(x_2+1), 0)
\end{aligned}
$$

so $\nabla g_1(1,0,0) = (0,-2,0)$ and $\nabla g_2(1,0,0) = (0,2,0)$.

Because $g_1(1,0,0) = 0$ and $g_2(1,0,0) = 0$, we know that $I(1,0,0) = \{1,2\}$, so we seek d such that $\nabla g_1(1,0,0)^T d < 0$ and $\nabla g_2(1,0,0)^T d < 0$. These reduce to $-2d_2 < 0$ and $2d_2 < 0$.

Clearly there exists no such d, so the Mangasarian-Fromowitz constraint qualification does not hold. We now show that it is also impossible to find Lagrange multipliers at this point, that is that the point x^* is not a KKT point.

First, observe that the Lagrangian is given by

$$\begin{aligned} L(x,\lambda) &= x_1 + x_2 + x_3^2 + \lambda_1((x_1-1)^2 \\ &+ (x_2-1)^2 - 1) + \lambda_2((x_1-1)^2 + (x_2+1)^2 - 1) \end{aligned}$$

and so

$$\nabla_x L(x,\lambda) = \begin{pmatrix} 1 + 2\lambda_1(x_1-1) + 2\lambda_2(x_1-1) \\ 1 + 2\lambda_1(x_2-1) + 2\lambda_2(x_2+1) \\ 2x_3 \end{pmatrix}.$$

Now suppose x^* *is* a KKT point. Then there must exist KKT multipliers λ such that $\nabla_x L(x^*, \lambda) = 0$. But

$$\nabla_x L((1,0,0), \lambda) = \begin{pmatrix} 1 \\ 1 - 2\lambda_1 + 2\lambda_2 \\ 0 \end{pmatrix}$$

which cannot equal $(0,0,0)$. So there cannot be such a λ, and x^* is not a KKT point.