620.370 Statistics for Mechanical Engineers — Semester 2, 2009

Homework set 7

Problems to be discussed at next week's tutorial: Quiz 7; 127, 129, 135, 130.

Homework questions

Questions 7.1–7.3 refer to the following information:

A random sample of 16 observations on $X \stackrel{d}{=} N(\mu, \sigma^2)$ gave $\bar{x} = 2.40$ and $s = 1.20$.

7.1 A 95% confidence interval for μ is:

[A.] $(1.874, 2.926)$; [B.] $(1.812, 2.988)$; [C.] $(1.767, 3.033)$;
[D.] $(1.764, 3.036)$; [E.] $(1.761, 3.039)$.

7.2 A 95% confidence interval for σ^2 is:

[A.] $(0.655, 2.874)$; [B.] $(0.749, 3.127)$; [C.] $(0.786, 3.449)$;
[D.] $(0.821, 2.713)$; [E.] $(0.864, 2.975)$.

7.3 A 95% prediction interval for X is:

[A.] $(-0.236, 5.036)$; [B.] $(-0.210, 5.010)$; [C.] $(-0.157, 4.957)$;
[D.] $(-0.024, 4.824)$; [E.] $(0.232, 4.568)$.

7.4 We obtain one observation on $Z \stackrel{d}{=} N(\theta, 1)$ (as, for example, with $Z = \dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$) and we wish to test H_0: $\theta = 0$ vs H_1: $\theta \neq 0$. We observe $z = -0.81$. The P-value is equal to

[A.] 0.810; [B.] 0.791; [C] 0.582; [D.] 0.418; [E.] 0.209; [F.] 0.190.

7.5 A random sample of $n = 5$ observations is obtained on $X \stackrel{d}{=} N(\mu, \sigma^2)$. To test H_0: $\sigma^2 = 1$ against H_1: $\sigma^2 > 1$, the critical region for a test of size 0.05 is:

[A.] $S^2 > 2.132$; [B.] $S^2 > 2.214$; [C.] $S^2 > 2.372$; [D.] $S^2 > 2.566$; [E.] $S^2 > 2.785$.

7.6 (a) A sequence of 20 independent trials resulted in 4 successes. If each trial has probability of success θ, show that the log-likelihood for these data is given by

$$\ln L(\theta) = \kappa + 4 \ln \theta + 16 \ln(1 - \theta) \quad (0 < \theta < 1).$$

i. Hence find the maximum likelihood estimate $\hat{\theta}$ and its standard error using the likelihood approach.

ii. Plot a graph of the relative log-likelihood function,

$$\text{RLL}(\theta) = \ln L(\theta) - \ln L(\hat{\theta});$$

and hence determine the likelihood-based 95% confidence interval.

iii. Compare this result with the exact interval obtained either from the Statistical Tables, or MATLAB.

(b) Suppose we wish to test H_0: $\theta = 0.5$ vs H_1: $\theta \neq 0.5$ on the basis of an observation on X, the number of successes obtained in 20 independent trials, each with probability of success θ.

i. If we observe $x = 4$, based on the confidence interval found in (a), is H_0 rejected or not?

ii. Evaluate $P = 2 \Pr(X' \leqslant 4)$, where $X' \stackrel{d}{=} \text{Bi}(20, 0.5)$, and indicate whether this indicates rejection or otherwise of the null hypothesis.

iii. Consider the test specified by the rule: "reject H_0 if $X \leqslant 5$ or if $X \geqslant 15$". Show that this test has size 0.041, and evaluate the power of this test when $\theta = 0.2$.

Quiz 7

Q7.1 A sequence of twenty independent trials results in 16 successes.
An exact 95% confidence interval for the probability of success is

[A.] $0.50 < p < 0.98$; [B.] $0.57 < p < 0.94$; [C.] $0.62 < p < 0.98$;
[D.] $0.65 < p < 0.94$; [E.] $0.78 < p < 0.82$.

Q7.2 A sequence of 100 independent trials each having probability of success p, yields 10 successes. Using an estimate of the variance of $\hat{p}$, an approx 95% confidence interval for p is given by:

[A.] $(0.041, 0.159)$; [B.] $(0.051, 0.149)$; [C.] $(0.081, 0.119)$;
[D.] $(0.094, 0.106)$; [E.] $(0.098, 0.102)$.

Q7.3 Let $X \stackrel{\mathrm{d}}{=} \mathrm{N}(\mu, 4)$ i.e. σ is known. A 95% confidence interval for μ based on a random sample of size 10 is found to be 4.2 ± 2.48. How large a sample would be required to give a 95% confidence interval of the form $\bar{x} \pm 1.24$?

[A.] 15; [B.] 20; [C.] 25; [D.] 30; [E.] 40.

Q7.4 To test H_0: $\theta = 0.4$ against H_1: $\theta = 0.8$, we use a test statistic, $T \stackrel{\mathrm{d}}{=} \mathrm{Bi}(10, \theta)$, and we reject H_0 if $T \geqslant 7$. Which one of the following statements is true?

[A.] $\text{size} = 0.055$, $\text{power} = 0.879$; [B.] $\text{size} = 0.055$, $\text{power} = 0.999$;
[C.] $\text{size} = 0.043$, $\text{power} = 0.879$; [D.] $\text{size} = 0.043$, $\text{power} = 0.999$;
[E.] $\text{size} = 0.012$, $\text{power} = 0.967$.

Q7.5 A significance test gives a P-value of 0.04 when testing a null hypothesis.
From this we would:

[A.] reject the hypothesis at the 0.01 level;
[B.] reject the hypothesis at the 0.05 level;
[C.] conclude that there is a probability of 0.04 that the null hypothesis is true;
[D.] conclude that there is a probability of 0.04 that the null hypothesis is false;
[E.] conclude that there is a probability of 0.04 that the data are wrong.