620-371: Linear Models

Assignment 1

Due: Monday, 30th March, 2009

This assignment is worth 5% of your total mark. Fill in a plagiarism declaration form and hand it in together with this assignment.

1. [4 marks] Let $A_1, A_2, \ldots, A_m$ be a collection of symmetric $k \times k$ matrices. Suppose that all A_i, $i = 1, 2, \ldots, m$ are idempotent, and that $A_i A_j = 0$ for all $i \neq j$. Show that $\sum_{i=1}^m A_i$ is an idempotent matrix.

   Solution:

$$\begin{aligned} \left(\sum_{i=1}^m A_i\right)\left(\sum_{i=1}^m A_i\right) &= \sum_{i=1}^m \sum_{j=1}^m A_i A_j \\ &= \sum_{i=1}^m A_i A_i \\ &= \sum_{i=1}^m A_i. \end{aligned}$$

2. Let

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, A = \frac{1}{6}\begin{bmatrix} 2 & -2 & 2 \\ -2 & 5 & 1 \\ 2 & 1 & 5 \end{bmatrix}, B = \begin{bmatrix} -2 & 5 & 3 \\ 5 & 1 & -4 \\ 3 & -4 & 0 \end{bmatrix}.$$

   Suppose that

$$E[\mathbf{y}] = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}, \text{var } \mathbf{y} = V = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

   (a) [2 marks] Find $E[\mathbf{y}^T A \mathbf{y}]$.

   Solution:

$$\begin{aligned} E[\mathbf{y}^T A \mathbf{y}] &= tr(AV) + \boldsymbol{\mu}^T A \boldsymbol{\mu} \\ &= tr\left(\frac{2}{3}\begin{bmatrix} 2 & -2 & 2 \\ -2 & 5 & 1 \\ 2 & 1 & 5 \end{bmatrix}\right) + \frac{1}{6}\begin{bmatrix} 1 & -3 & -2 \end{bmatrix}\begin{bmatrix} 2 & -2 & 2 \\ -2 & 5 & 1 \\ 2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} \\ &= 8 + \frac{83}{6} = \frac{131}{6}. \end{aligned}$$

(b) [4 marks] Describe the distribution of $\frac{1}{4}\mathbf{y}^T A\mathbf{y}$.

Solution: A is symmetric and idempotent and $r(A) = 2$, so $\frac{1}{4}\mathbf{y}^T A\mathbf{y}$ has a noncentral χ^2 distribution with 2 degrees of freedom and noncentrality parameter $\frac{1}{8}\boldsymbol{\mu}^T A\boldsymbol{\mu} = \frac{83}{48}$.

(c) [2 marks] Are $\mathbf{y}^T A\mathbf{y}$ and $\mathbf{y}^T B\mathbf{y}$ independent?

Solution: $AVB \neq 0$, so $\mathbf{y}^T A\mathbf{y}$ and $\mathbf{y}^T B\mathbf{y}$ are not independent.

3. A study is conducted to determine if (and how) the fuel mileage of a car is dependent on its weight, and the speed at which it is driven. A linear model is assumed, and the following data is obtained:

Mileage (km/litre)	Weight (tons)	Speed (km/hr)
8.5	1.35	34
8	1.33	36
7.5	2	38
10	1.4	34
11	1.4	31
15	1.2	31
13.5	1.3	33
14.5	1.28	41

(a) [2 marks] Write down the linear model in matrix form.

Solution: $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, where

$$\mathbf{y} = \begin{bmatrix} 8.5 \\ 8 \\ 7.5 \\ 10 \\ 11 \\ 15 \\ 13.5 \\ 14.5 \end{bmatrix}, X = \begin{bmatrix} 1 & 1.35 & 34 \\ 1 & 1.33 & 36 \\ 1 & 2 & 38 \\ 1 & 1.4 & 34 \\ 1 & 1.4 & 31 \\ 1 & 1.2 & 31 \\ 1 & 1.3 & 33 \\ 1 & 1.28 & 41 \end{bmatrix}, \boldsymbol{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}, \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \\ \varepsilon_7 \\ \varepsilon_8 \end{bmatrix}.$$

(b) [2 marks] Write down the normal equations for this model.

Solution:

$$(X^T X)\boldsymbol{\beta} = \begin{bmatrix} 8 & 11.3 & 278 \\ 11.3 & 16.3 & 393.3 \\ 278 & 393.3 & 9744 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = X^T\mathbf{y} = \begin{bmatrix} 88 \\ 120.6 \\ 3048 \end{bmatrix}.$$

(c) [2 marks] Solve the normal equations to estimate the parameters.

Solution:

$$\boldsymbol{\beta} = (X^T X)^{-1} X^T \mathbf{y} = \begin{bmatrix} 19.44 \\ -7.86 \\ 0.08 \end{bmatrix}.$$

You may use R for this question, but for the matrix calculations only. If you do, include your R commands and output.