

620-371: Linear Models

Practice Class 2

10th March, 2009

1. Show that $X^T X$ is a symmetric matrix.

Solution:

$$(X^T X)^T = X^T (X^T)^T = X^T X.$$

2. (a) Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be a nonsingular 2×2 matrix. Show by direct multiplication that

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Solution:

$$\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} da - bc & db - bd \\ -ca + ac & -cb + ad \end{bmatrix} = I.$$

- (b) Find the inverse of

$$\begin{bmatrix} 2 & 4 \\ 1 & -3 \end{bmatrix}.$$

Solution: From above, the inverse is

$$-\frac{1}{10} \begin{bmatrix} -3 & -4 \\ -1 & 2 \end{bmatrix}.$$

3. Is

$$X = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

orthogonal? If not, what value of c makes the matrix cX orthogonal?

Solution: The matrix is not orthogonal, as its columns do not form an orthonormal set (e.g. the first column has norm > 1). However they do form an *orthogonal* set, so we can just normalise each vector to produce an orthogonal matrix. This gives $c = \frac{1}{2}$.

4. (a) Find the eigenvalues, and an associated eigenvector for each eigenvalue, of the matrix

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

Solution: The characteristic equation is

$$\det(A - \lambda I) = (2 - \lambda)^2 - 4 = -4\lambda + \lambda^2 = \lambda(\lambda - 4) = 0,$$

so the eigenvalues are 0 and 4.

For $\lambda = 4$, we solve

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

One such solution is

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For $\lambda = 0$, we solve

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

One such solution is

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

- (b) Find an orthogonal matrix P such that $P^T A P$ is diagonal.

Solution:

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

- (c) Write down $P^T A P$ for the P given in part (b).

Solution:

$$P^T A P = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}.$$

5. Let

$$A = \begin{bmatrix} 1 & 4 & 3 \\ -2 & 0 & 2 \\ 4 & 4 & 0 \end{bmatrix}.$$

- (a) Write down the trace of A .

Solution: $\text{tr}(A) = 1$.

- (b) Are the columns of A linearly independent? Justify your answer.

Solution: The columns of A are not linearly independent — the second column is the sum of the first and third.

(c) Find the rank of A .

Solution: The first and third columns of A are not multiples of each other, so $r(A) = 2$.

6. Show that if X is of full rank, then

$$I - X(X^T X)^{-1} X^T$$

is an idempotent matrix.

Solution: It was shown that $A = X(X^T X)^{-1} X^T$ is idempotent. Therefore

$$(I - A)(I - A) = I - A - A + A^2 = I - A - A + A = I - A$$

is idempotent.