620-371: Linear Models

Practice Class 3

17th March, 2009

1. Let X be a 10×5 matrix of full rank. We know that $H = X(X^T X)^{-1} X^T$ is idempotent. It can also be shown that it is symmetric.
   (a) Find $tr(H)$.
   (b) Find $r(H)$.

2. Let
$$A = \begin{bmatrix} 3 & 1 & 8 \\ 1 & 0 & 1 \\ 2 & 1 & -4 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$
   (a) Let $z = \mathbf{y}^T A \mathbf{y}$. Write out z in full.
   (b) Let $z = \mathbf{y}^T A \mathbf{y}$. Find $\frac{\partial z}{\partial \mathbf{y}}$.

3. Let $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T$ be a random vector with mean $\boldsymbol{\mu} = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}^T$, and assume that
$$\text{var } y_i = 4, \text{cov}(y_i, y_j) = 0.$$
   (a) Write down var $\mathbf{y}$.
   (b) Let
$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \\ -1 & 6 & 1 \end{bmatrix}.$$
   Find var $A\mathbf{y}$.
   (c) Find $E[\mathbf{y}^T A \mathbf{y}]$.

4. Prove the second corollary on lecture slide 32 of 'Random vectors': If $\mathbf{y}$ is an $n \times 1$ normally distributed random vector with mean $\boldsymbol{\mu}$ and variance $\sigma^2 I$, then show that $\frac{1}{\sigma^2}\mathbf{y}^T A \mathbf{y}$ has a noncentral χ^2 distribution with k degrees of freedom and noncentrality parameter $\lambda = \frac{1}{2\sigma^2}\boldsymbol{\mu}^T A \boldsymbol{\mu}$ if and only if A is idempotent and has rank k.

5. Let $\mathbf{y} = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T$ be a normal random vector with mean and variance
$$\boldsymbol{\mu} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$
   Let
$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, B = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

(a) Find the distributions of $\mathbf{y}^T A\mathbf{y}$ and $\mathbf{y}^T B\mathbf{y}$.

(b) Are $\mathbf{y}^T A\mathbf{y}$ and $\mathbf{y}^T B\mathbf{y}$ independent?

(c) What is the distribution of $\mathbf{y}^T A\mathbf{y} + \mathbf{y}^T B\mathbf{y}$?