1. Let $X$ be a $10 \times 5$ matrix of full rank. We know that $H = X(X^TX)^{-1}X^T$ is idempotent. It can also be shown that it is symmetric.

   (a) Find $tr(H)$.
   (b) Find $r(H)$.

2. Let 
\[
A = \begin{bmatrix}
3 & 1 & 8 \\
1 & 0 & 1 \\
2 & 1 & -4
\end{bmatrix},
\] 
$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$.

   (a) Let $z = y^TAy$. Write out $z$ in full.
   (b) Let $z = y^TAy$. Find $\frac{\partial z}{\partial y}$.

3. Let $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}^T$ be a random vector with mean $\mu = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}^T$, and assume that $\text{var} \ y_i = 4, \text{cov}(y_i, y_j) = 0$.

   (a) Write down $\text{var} \ y$.
   (b) Let 
\[
A = \begin{bmatrix}
2 & -3 & 1 \\
1 & 2 & 0 \\
-1 & 6 & 1
\end{bmatrix}.
\]

   Find $\text{var} \ Ay$.
   (c) Find $E[y^TAy]$.

4. Prove the second corollary on lecture slide 32 of 'Random vectors': If $y$ is an $n \times 1$ normally distributed random vector with mean $\mu$ and variance $\sigma^2 I$, then show that $\frac{1}{\sigma^2} y^TAy$ has a noncentral $\chi^2$ distribution with $k$ degrees of freedom and noncentrality parameter $\lambda = \frac{1}{\sigma^2} \mu^T A \mu$ if and only if $A$ is idempotent and has rank $k$.

5. Let $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^T$ be a normal random vector with mean and variance 
\[
\mu = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \ V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

Let 
\[
A = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \ B = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.
\]
(a) Find the distributions of $y^T A y$ and $y^T B y$.
(b) Are $y^T A y$ and $y^T B y$ independent?
(c) What is the distribution of $y^T A y + y^T B y$?