620-371: Linear Models

Practice Class 3

17th March, 2009

1. Let X be a 10×5 matrix of full rank.

   (a) Find $tr(H)$.
   Solution: $tr(H) = tr(X(X^TX)^{-1}X^T) = tr(X^TX(X^TX)^{-1}) = tr(I_5) = 5$.

   (b) Find $r(H)$.
   Solution: Since H is symmetric and idempotent, $r(H) = tr(H) = 5$.

2. Let

$$A = \begin{bmatrix} 3 & 1 & 8 \\ 1 & 0 & 1 \\ 2 & 1 & -4 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

   and let $z = \mathbf{y}^T A\mathbf{y}$.

   (a) Write out z in full.
   Solution: $z = 3y_1^2 - 4y_3^2 + 2y_1y_2 + 10y_1y_3 + 2y_2y_3$.

   (b) Find $\frac{\partial z}{\partial \mathbf{y}}$.
   Solution:

$$\frac{\partial z}{\partial \mathbf{y}} = A\mathbf{y} + A^T\mathbf{y} = \begin{bmatrix} 6y_1 + 2y_2 + 10y_3 \\ 2y_1 + 2y_3 \\ 10y_1 + 2y_2 - 8y_3 \end{bmatrix}.$$

3. Let $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix}^T$ be a random vector with mean $\boldsymbol{\mu} = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}^T$, and assume that

$$\text{var } y_i = 4, \text{cov}(y_i, y_j) = 0.$$

   (a) Write down var $\mathbf{y}$.
   Solution:

$$\text{var } \mathbf{y} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

   (b) Let

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \\ -1 & 6 & 1 \end{bmatrix}.$$

   Find var $A\mathbf{y}$.

Solution:

$$\begin{aligned}
\text{var } A\mathbf{y} &= AVA^T = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \\ -1 & 6 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ -3 & 2 & 6 \\ 1 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \\ -1 & 6 & 1 \end{bmatrix} \begin{bmatrix} 8 & 4 & -4 \\ -12 & 8 & 24 \\ 4 & 0 & 4 \end{bmatrix} \\
&= \begin{bmatrix} 56 & -16 & 76 \\ -16 & 20 & 44 \\ -76 & 44 & 152 \end{bmatrix}.
\end{aligned}$$

(c) Find $E[\mathbf{y}^T A\mathbf{y}]$.

Solution:

$$\begin{aligned}
E[\mathbf{y}^T A\mathbf{y}] &= tr(AV) + \boldsymbol{\mu}^T A\boldsymbol{\mu} \\
&= tr\left(\begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \\ -1 & 6 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}\right) + \begin{bmatrix} 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & 0 \\ -1 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \\
&= tr\left(\begin{bmatrix} 8 & -12 & 4 \\ 4 & 8 & 0 \\ -4 & 24 & 4 \end{bmatrix}\right) + \begin{bmatrix} 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} -5 \\ 7 \\ 19 \end{bmatrix} \\
&= 20 + 54 = 74.
\end{aligned}$$

4. Prove the second corollary on lecture slide 32 of 'Random vectors': If $\mathbf{y}$ is an $n \times 1$ normally distributed random vector with mean $\boldsymbol{\mu}$ and variance $\sigma^2 I$, then show that $\frac{1}{\sigma^2}\mathbf{y}^T A\mathbf{y}$ has a noncentral χ^2 distribution with k degrees of freedom and noncentrality parameter $\lambda = \frac{1}{2\sigma^2}\boldsymbol{\mu}^T A\boldsymbol{\mu}$ if and only if A is idempotent and has rank k.

   Solution: From the distribution of $\mathbf{y}$, we know that $\frac{1}{\sigma}\mathbf{y}$ is a normally distributed random vector with mean $\frac{1}{\sigma}\boldsymbol{\mu}$ and variance I. Therefore $(\frac{1}{\sigma}\mathbf{y})^T A(\frac{1}{\sigma}\mathbf{y}) = \frac{1}{\sigma^2}\mathbf{y}^T A\mathbf{y}$ has a noncentral χ^2 distribution with k degrees of freedom and noncentrality parameter

$$\lambda = \frac{1}{2}(\frac{1}{\sigma}\boldsymbol{\mu})^T A(\frac{1}{\sigma}\boldsymbol{\mu}) = \frac{1}{2\sigma^2}\boldsymbol{\mu}^T A\boldsymbol{\mu}$$

   if and only if A is idempotent and has rank k.

5. Let $\mathbf{y} = \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T$ be a normal random vector with mean and variance

$$\boldsymbol{\mu} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

   Let

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, B = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

(a) Find the distributions of $\mathbf{y}^T A\mathbf{y}$ and $\mathbf{y}^T B\mathbf{y}$.

Solution: A and B are both idempotent and have rank 1, so $\mathbf{y}^T A\mathbf{y}$ and $\mathbf{y}^T B\mathbf{y}$ have noncentral χ^2 distributions with 1 degree of freedom each and noncentrality parameters

$$\frac{1}{2}\begin{bmatrix} 2 & 4 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 9$$

and

$$\frac{1}{2}\begin{bmatrix} 2 & 4 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 1$$

respectively.

(b) Are $\mathbf{y}^T A\mathbf{y}$ and $\mathbf{y}^T B\mathbf{y}$ independent?

Solution: $AB = 0$, so they are independent.

(c) What is the distribution of $\mathbf{y}^T A\mathbf{y} + \mathbf{y}^T B\mathbf{y}$?

Solution: $\mathbf{y}^T A\mathbf{y} + \mathbf{y}^T B\mathbf{y}$ has a noncentral χ^2 distribution with 2 degrees of freedom and noncentrality parameter 10.