620-371: Linear Models

Practice Class 8

28th April, 2009

1. It is known that $R(\boldsymbol{\gamma}_1|\boldsymbol{\gamma}_2)$ has a noncentral χ^2 distribution with r degrees of freedom and noncentrality parameter

$$\lambda = \frac{1}{2\sigma^2}\boldsymbol{\beta}^T X^T[X(X^TX)^{-1}X^T - X_2(X_2^TX_2)^{-1}X_2^T]X\boldsymbol{\beta}.$$

Show that if $H_0 : \boldsymbol{\gamma}_1 = \mathbf{0}$ is true, then $\lambda = 0$. (*Hint: Partition X and $\boldsymbol{\beta}$ and calculate $X\boldsymbol{\beta}$*).

Solution: From partitioning,

$$\begin{aligned}
X\boldsymbol{\beta} &= \left[\frac{X_1}{X_2}\right]\left[\ \boldsymbol{\gamma}_1 \mid \boldsymbol{\gamma}_2\ \right] \\
&= X_1\boldsymbol{\gamma}_1 + X_2\boldsymbol{\gamma}_2 \\
&= X_2\boldsymbol{\gamma}_2.
\end{aligned}$$

This gives

$$\begin{aligned}
2\sigma^2\lambda &= \boldsymbol{\beta}^T X^T X(X^TX)^{-1}X^TX\boldsymbol{\beta} - \boldsymbol{\beta}^TX^TX_2(X_2^TX_2)^{-1}X_2^TX\boldsymbol{\beta} \\
&= \boldsymbol{\beta}^TX^TX\boldsymbol{\beta} - \boldsymbol{\gamma}_2^TX_2^TX_2(X_2^TX_2)^{-1}X_2^TX_2\boldsymbol{\gamma}_2 \\
&= \boldsymbol{\gamma}_2^TX_2^TX_2\boldsymbol{\gamma}_2 - \boldsymbol{\gamma}_2^TX_2^TX_2\boldsymbol{\gamma}_2 \\
&= 0.
\end{aligned}$$

2. In this question, we will show (in a different way to the lecture slides) that $(\mathbf{b}-\boldsymbol{\beta}^*)^TX^TX(\mathbf{b}-\boldsymbol{\beta}^*)$ and SS_{Res} are independent. We set $\mathbf{q} = \mathbf{y} - X\boldsymbol{\beta}^*$.

   (a) Show that $(\mathbf{b}-\boldsymbol{\beta}^*)^TX^TX(\mathbf{b}-\boldsymbol{\beta}^*) = \mathbf{q}^TX(X^TX)^{-1}X^T\mathbf{q}$.
   Solution:

$$\begin{aligned}
\mathbf{q}^TX(X^TX)^{-1}X^T\mathbf{q} &= \mathbf{y}^TX(X^TX)^{-1}X^T\mathbf{y} - (\boldsymbol{\beta}^*)^TX^TX(X^TX)^{-1}X^T\mathbf{y} \\
&\quad -\mathbf{y}^TX(X^TX)^{-1}X^TX\boldsymbol{\beta}^* + (\boldsymbol{\beta}^*)^TX^TX(X^TX)^{-1}X^TX\boldsymbol{\beta}^* \\
&= \mathbf{b}^TX^TX\mathbf{b} - (\boldsymbol{\beta}^*)^TX^TX\mathbf{b} - \mathbf{b}^TX^TX\boldsymbol{\beta}^* + (\boldsymbol{\beta}^*)^TX^TX\boldsymbol{\beta}^* \\
&= (\mathbf{b}-\boldsymbol{\beta}^*)^TX^TX(\mathbf{b}-\boldsymbol{\beta}^*).
\end{aligned}$$

(b) It can be shown that $SS_{Res} = \mathbf{q}^T[I - X(X^TX)^{-1}X^T]\mathbf{q}$. Show that these two quadratic forms are independent.

Solution: We know that var $\mathbf{q} = \sigma^2 I$, so

$$\begin{aligned} AVB &= X(X^TX)^{-1}X^T\sigma^2 I[I - X(X^TX)^{-1}X^T] \\ &= \sigma^2\left(X(X^TX)^{-1}X^T - X(X^TX)^{-1}X^TX(X^TX)^{-1}X^T\right) \\ &= \sigma^2\left(X(X^TX)^{-1}X^T - X(X^TX)^{-1}X^T\right) \\ &= 0. \end{aligned}$$

3. Consider the hypothesis $H_0 : \boldsymbol{\beta} = \boldsymbol{\beta}^*$. Formulate this hypothesis in terms of the general linear hypothesis and show that the test for this general linear hypothesis is equivalent to nonzero test given earlier in the slides.

   Solution: We have $C = I_p$ and $\boldsymbol{\delta}^* = \boldsymbol{\beta}^*$. The F statistic for the general linear hypothesis is

$$\frac{(C\mathbf{b} - \boldsymbol{\delta}^*)^T[C(X^TX)^{-1}C^T]^{-1}(C\mathbf{b} - \boldsymbol{\delta}^*)/r}{SS_{Res}/(n-p)} = \frac{(\mathbf{b} - \boldsymbol{\beta}^*)^TX^TX(\mathbf{b} - \boldsymbol{\beta}^*)/p}{SS_{Res}/(n-p)},$$

   which is the same statistic as the one used for the nonzero test.

4. Suppose we have a linear model with 6 parameters and we want to simultaneously test the hypotheses $\beta_0 = 1$, $\beta_1 = \beta_2 = 2\beta_3 - 2$, $\beta_4 - \beta_1 = \beta_5$. Write down the C matrix and $\boldsymbol{\delta}^*$ vector for the general linear hypothesis which tests these.

   Solution:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & -1 \end{bmatrix}, \boldsymbol{\delta}^* = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}.$$

5. Consider the model

$$y = \beta_0 + \beta_1(x_1 - \bar{x}_1) + \beta_2(x_2 - \bar{x}_2) + \varepsilon.$$

   An experiment is designed and run with the following data:

x_1	x_2	y
10	50	25
100	50	29
10	100	30
100	100	40

(a) Write down the X matrix.

Solution:

$$X = \begin{bmatrix} 1 & -45 & -25 \\ 1 & 45 & -25 \\ 1 & -45 & 25 \\ 1 & 45 & 25 \end{bmatrix}.$$

(b) Calculate $R(\beta_0|\beta_1, \beta_2)$ and $R(\beta_1|\beta_0, \beta_2)$.

Solution:

$$\begin{aligned} SS_{Reg} &= \mathbf{y}^T X(X^T X)^{-1} X^T \mathbf{y} = 3957 \\ R(\beta_1, \beta_2) &= 113 \\ R(\beta_0|\beta_1, \beta_2) &= 3844 \\ R(\beta_0, \beta_2) &= 3908 \\ R(\beta_1|\beta_0, \beta_2) &= 49 \end{aligned}$$

(c) Calculate $R(\beta_0)$ and $R(\beta_1|\beta_0)$.

Solution:

$$\begin{aligned} R(\beta_0) &= 3844 \\ R(\beta_1|\beta_0) &= R(\beta_0, \beta_1) - R(\beta_0) = 49 \end{aligned}$$

(d) Show that this model is mutually orthogonal, i.e. the columns of X are orthogonal to each other.

Solution: It is apparent by inspection (but can be easily calculated) that the inner product of any two columns of X is 0.

(e) Test the hypothesis $H_0 : \beta_0 + \beta_1 + \beta_2 = 35$. You may take the critical value at 5% for an F distribution with 1 and 1 degree(s) of freedom to be 161.45.

Solution: We have

$$C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \boldsymbol{\delta}^* = \begin{bmatrix} 35 \end{bmatrix}.$$

The rank of C is 1, so we use the F-statistic

$$\frac{(C\mathbf{b} - \boldsymbol{\delta}^*)^T [C(X^T X)^{-1} C^T]^{-1} (C\mathbf{b} - \boldsymbol{\delta}^*)/1}{SS_{Res}/(n-p)} = 6.28.$$

Since $6.28 < 161.45$, we cannot reject the null hypothesis.