1. Show that in a mutually orthogonal full rank model, $X^T X$ is diagonal. Calculate $(X^T X)^{-1}$ (expressed in terms of the elements of the $X$ matrix).

2. Consider a less than full rank model with two factors. Factor 1 has two levels, and factor 2 has 3 levels. We take 2 samples from each possible combination of factor levels. We denote the response variable from the $k$th sample from the combination of factors with the first factor at level $i$ and the second factor at level $j$ to be $y_{ijk}$. We also denote the overall mean by $\mu$, and assume that each level of each factor adjusts this overall mean by a certain amount: $\tau_i$ for the $i$th level of factor 1, and $\beta_j$ for the $j$th level of factor 2.

   (a) Express $y_{ijk}$ according to $\mu$, $\tau_i$, $\beta_j$, and an error term.
   (b) Write down the linear model in matrix form.

3. Let

\[
A = \begin{bmatrix}
1 & 2 & 5 & 2 \\
3 & 7 & 12 & 4 \\
0 & 1 & -3 & -2
\end{bmatrix}.
\]

   (a) Show that $r(A) = 2$.
   (b) Find a conditional inverse for $A$.

4. Show that $A = A(A^T A)^c A^T A$. You may use the result that if $A^T A = 0$, then $A = 0$. (Hint: Consider the matrix $A - A(A^T A)^c A^T A$).

5. It is known that toxic material was dumped into a river that flows into a large salt-water commercial fishing area. We are interested in the amount of toxic material (in parts per million) found in oysters harvested at three different locations in this area. A study is conducted and the following data obtained:

\[
\begin{array}{ccc}
\text{Site 1} & \text{Site 2} & \text{Site 3} \\
15 & 19 & 22 \\
26 & 15 & 26
\end{array}
\]

   (a) Write down the linear model in matrix form.
   (b) Write down the normal equations.
   (c) Find a conditional inverse for $X^T X$.
   (d) Find a solution for the normal equations.