1. Show that in a mutually orthogonal full rank model, $X^T X$ is diagonal. Calculate $(X^T X)^{-1}$ (expressed in terms of the elements of the $X$ matrix).

**Solution:** The $(i,j)$th element of $X^T X$ is

$$\sum_k X_{ki} X_{kj} = X_i^T X_j$$

where $X_i$ and $X_j$ are the $i$th and $j$th columns of $X$ respectively. Since the model is mutually orthogonal, this is 0 if $i \neq j$ and therefore $X^T X$ is diagonal. The $(i,i)$th element of $X^T X$ is

$$\sum_k X_{ki} X_{ki} = \|X_i\|^2.$$

Therefore $(X^T X)^{-1}$ is the $p \times p$ diagonal matrix with the $i$th diagonal entry $\|X_i\|^{-2}$.

2. Consider a less than full rank model with two factors. Factor 1 has two levels, and factor 2 has 3 levels. We take 2 samples from each possible combination of factor levels. We denote the response variable from the $k$th sample from the combination of factors with the first factor at level $i$ and the second factor at level $j$ to be $y_{ijk}$. We also denote the overall mean by $\mu$, and assume that each level of each factor adjusts this overall mean by a certain amount: $\tau_i$ for the $i$th level of factor 1, and $\beta_j$ for the $j$th level of factor 2.

(a) Express $y_{ijk}$ according to $\mu$, $\tau_i$, $\beta_j$, and an error term.

**Solution:** $y_{ijk} = \mu + \tau_i + \beta_j + \varepsilon_{ijk}$, for $i = 1, 2$, $j = 1, 2, 3$, and $k = 1, 2$.

(b) Write down the linear model in matrix form.
Solution: $y = X\beta + \varepsilon$, where

\[
y = \begin{bmatrix}
y_{111} \\
y_{112} \\
y_{121} \\
y_{122} \\
y_{131} \\
y_{132} \\
y_{211} \\
y_{212} \\
y_{221} \\
y_{222} \\
y_{231} \\
y_{232}
\end{bmatrix},
X = \begin{bmatrix}
1 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1
\end{bmatrix},
\beta = \begin{bmatrix}
\mu \\
\tau_1 \\
\tau_2 \\
\beta_1 \\
\beta_2 \\
\beta_3
\end{bmatrix},
\varepsilon = \begin{bmatrix}
e_{111} \\
e_{112} \\
e_{121} \\
e_{122} \\
e_{131} \\
e_{132} \\
e_{211} \\
e_{212} \\
e_{221} \\
e_{222} \\
e_{231} \\
e_{232}
\end{bmatrix}.
\]

3. Let

\[
A = \begin{bmatrix}
1 & 2 & 5 & 2 \\
3 & 7 & 12 & 4 \\
0 & 1 & -3 & -2
\end{bmatrix}.
\]

(a) Show that $r(A) = 2$.

**Solution:** It is easy to see that the third row of $A$ is the second row minus 3 times the first row, but the first two rows are linearly independent. Therefore $r(A) = 2$.

(b) Find a conditional inverse for $A$.

**Solution:** The inverse of the top left $2 \times 2$ minor of $A$ is

\[
\begin{bmatrix}
7 & -2 \\
-3 & 1
\end{bmatrix},
\]

so using the conditional inverse algorithm gives

\[
A^c = \begin{bmatrix}
7 & -2 & 0 \\
-3 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

4. Show that $A = A(A^TA)^cA^TA$. You may use the result that if $A^TA = 0$, then $A = 0$. (**Hint:** Consider the matrix $A - A(A^TA)^cA^TA$).

**Solution:**

\[
(A - A(A^TA)^cA^TA)^T(A - A(A^TA)^cA^TA)
= A^TA - A^TA A(A^TA)^cA^TA - A^TA + A^TA A(A^TA)^cA^TA
= 0.
\]

Therefore $A = A(A^TA)^cA^TA = 0$ and $A = A(A^TA)^cA^TA$. 
5. It is known that toxic material was dumped into a river that flows into a large salt-water commercial fishing area. We are interested in the amount of toxic material (in parts per million) found in oysters harvested at three different locations in this area. A study is conducted and the following data obtained:

<table>
<thead>
<tr>
<th>Site 1</th>
<th>Site 2</th>
<th>Site 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>19</td>
<td>22</td>
</tr>
<tr>
<td>26</td>
<td>15</td>
<td>26</td>
</tr>
</tbody>
</table>

(a) Write down the linear model in matrix form.

**Solution:** \( y = X\beta + \varepsilon \), where

\[
\begin{bmatrix}
15 \\
26 \\
19 \\
15 \\
22 \\
26
\end{bmatrix},
\begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
\mu \\
\tau_1 \\
\tau_2 \\
\tau_3
\end{bmatrix},
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{12} \\
\varepsilon_{21} \\
\varepsilon_{22} \\
\varepsilon_{31} \\
\varepsilon_{32}
\end{bmatrix}.
\]

(b) Write down the normal equations.

**Solution:** \( X^T X \beta = X^T y \), where

\[
X^T X = \begin{bmatrix}
6 & 2 & 2 & 2 \\
2 & 2 & 0 & 0 \\
2 & 0 & 2 & 0 \\
2 & 0 & 0 & 2
\end{bmatrix},
X^T y = \begin{bmatrix}
123 \\
41 \\
34 \\
48
\end{bmatrix}.
\]

(c) Find a conditional inverse for \( X^T X \).

**Solution:**

\[(X^T X)^{c} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & \frac{1}{2}
\end{bmatrix}.
\]

(d) Find a solution for the normal equations.

**Solution:**

\[ b = (X^T X)^{c} X^T y = \begin{bmatrix}
0 \\
20.5 \\
17 \\
24
\end{bmatrix}. \]