620-371: Linear Models

Practice Class 9

5th April, 2009

1. Show that in a mutually orthogonal full rank model, $X^T X$ is diagonal. Calculate $(X^T X)^{-1}$ (expressed in terms of the elements of the X matrix).

   Solution: The (i, j)th element of $X^T X$ is

   $$\sum_k X_{ki} X_{kj} = X_i^T X_j$$

   where X_i and X_j are the ith and jth columns of X respectively. Since the model is mutually orthogonal, this is 0 if $i \neq j$ and therefore $X^T X$ is diagonal. The (i, i)th element of $X^T X$ is

   $$\sum_k X_{ki} X_{ki} = \|X_i\|^2.$$

   Therefore $(X^T X)^{-1}$ is the $p \times p$ diagonal matrix with the ith diagonal entry $\|X_i\|^{-2}$.

2. Consider a less than full rank model with two factors. Factor 1 has two levels, and factor 2 has 3 levels. We take 2 samples from each possible combination of factor levels. We denote the response variable from the kth sample from the combination of factors with the first factor at level i and the second factor at level j to be y_{ijk}. We also denote the overall mean by μ, and assume that each level of each factor adjusts this overall mean by a certain amount: τ_i for the ith level of factor 1, and β_j for the jth level of factor 2.

   (a) Express y_{ijk} according to μ, τ_i, β_j, and an error term.

   Solution: $y_{ijk} = \mu + \tau_i + \beta_j + \varepsilon_{ijk}$, for $i = 1, 2,\ j = 1, 2, 3$, and $k = 1, 2$.

   (b) Write down the linear model in matrix form.

Solution: $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, where

$$\mathbf{y} = \begin{bmatrix} y_{111} \\ y_{112} \\ y_{121} \\ y_{122} \\ y_{131} \\ y_{132} \\ y_{211} \\ y_{212} \\ y_{221} \\ y_{222} \\ y_{231} \\ y_{232} \end{bmatrix}, X = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \boldsymbol{\beta} = \begin{bmatrix} \mu \\ \tau_1 \\ \tau_2 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}, \boldsymbol{\varepsilon} = \begin{bmatrix} e_{111} \\ e_{112} \\ e_{121} \\ e_{122} \\ e_{131} \\ e_{132} \\ e_{211} \\ e_{212} \\ e_{221} \\ e_{222} \\ e_{231} \\ e_{232} \end{bmatrix}.$$

3. Let

$$A = \begin{bmatrix} 1 & 2 & 5 & 2 \\ 3 & 7 & 12 & 4 \\ 0 & 1 & -3 & -2 \end{bmatrix}.$$

   (a) Show that $r(A) = 2$.

   Solution: It is easy to see that the third row of A is the second row minus 3 times the first row, but the first two rows are linearly independent. Therefore $r(A) = 2$.

   (b) Find a conditional inverse for A.

   Solution: The inverse of the top left 2×2 minor of A is

$$\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix},$$

   so using the conditional inverse algorithm gives

$$A^c = \begin{bmatrix} 7 & -2 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

4. Show that $A = A(A^TA)^cA^TA$. You may use the result that if $A^TA = 0$, then $A = 0$. (*Hint: Consider the matrix* $A - A(A^TA)^cA^TA$).

   Solution:

$$\begin{aligned} &(A - A(A^TA)^cA^TA)^T(A - A(A^TA)^cA^TA) \\ &= A^TA - A^TA[(A^TA)^c]^TA^TA - A^TA(A^TA)^cA^TA + A^TA[(A^TA)^c]^TA^TA(A^TA)^cA^TA \\ &= A^TA - A^TA(A^TA)^cA^TA - A^TA + A^TA(A^TA)^cA^TA \\ &= 0. \end{aligned}$$

   Therefore $A - A(A^TA)^cA^TA = 0$ and $A = A(A^TA)^cA^TA$.

5. It is known that toxic material was dumped into a river that flows into a large salt-water commercial fishing area. We are interested in the amount of toxic material (in parts per million) found in oysters harvested at three different locations in this area. A study is conducted and the following data obtained:

Site 1	Site 2	Site 3
15	19	22
26	15	26

(a) Write down the linear model in matrix form.
Solution: $\mathbf{y} = X\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, where

$$\mathbf{y} = \begin{bmatrix} 15 \\ 26 \\ 19 \\ 15 \\ 22 \\ 26 \end{bmatrix}, X = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \boldsymbol{\beta} = \begin{bmatrix} \mu \\ \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix}, \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{12} \\ \varepsilon_{21} \\ \varepsilon_{22} \\ \varepsilon_{31} \\ \varepsilon_{32} \end{bmatrix}.$$

(b) Write down the normal equations.
Solution: $X^T X\mathbf{b} = X^T\mathbf{y}$, where

$$X^T X = \begin{bmatrix} 6 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}, X^T\mathbf{y} = \begin{bmatrix} 123 \\ 41 \\ 34 \\ 48 \end{bmatrix}.$$

(c) Find a conditional inverse for $X^T X$.
Solution:

$$(X^T X)^c = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

(d) Find a solution for the normal equations.
Solution:

$$\mathbf{b} = (X^T X)^c X^T\mathbf{y} = \begin{bmatrix} 0 \\ 20.5 \\ 17 \\ 24 \end{bmatrix}.$$