1. a) $\bar{y}_h = \frac{1}{n_h} \sum_{i=1}^{N_h} Y_{h,i} I_{h,i}$ ✓

$\mathbb{E}\,\bar{y}_h = \frac{1}{n_h} \sum_{i=1}^{N_h} Y_{h,i}\, \mathbb{E} I_{h,i}$; $\mathbb{E} I_{h,i} = \frac{n_h}{N_h}$ ✓

$= \frac{1}{N_h} \sum_{i=1}^{N_h} Y_{h,i} = \mu_h$ ✓

b) $\mathbb{E}\hat{\mu}_1 = \sum_{h=1}^{L} \frac{N_h}{N} \mu_h = \sum_{h=1}^{L} \sum_{i=1}^{N_h} Y_{h,i}/N = \mu$ ✓

$\mathbb{E}\hat{\mu}_2 \neq \mu$ in general (unless $\frac{n_h}{n} = \frac{N_h}{N}$) ✓

so prefer $\hat{\mu}$ as it is unbiased. ✓

c) use lagrange multiplier

$\min_{\underset{\sim}{n}}$ $Var\,\hat{\mu}_1 = \sum_{h=1}^{L} W_h^2 S_h^2 (1-f_h)/n_h$ (from formulas) ✓

s.t. $\sum_{h=1}^{L} c_h n_h^2 = C$

equiv. to

$\min_{\underset{\sim}{n},\lambda}$ $\sum_h W_h^2 S_h^2 \left(\frac{1}{n_h} - \frac{1}{N_h}\right) + \lambda\left(\sum_h c_h n_h^2 - C\right)$ ✓✓

$\frac{\partial}{\partial n_h} - " - = -\frac{W_h^2 S_h^2}{n_h^2} + 2\lambda c_h n_h$

$= 0 \implies \lambda = \frac{W_h^2 S_h^2}{2 c_h n_h^3}$ ✓

this is true for all h, so we have ✓✓

$n_h^3 \propto \frac{W_h^2 S_h^2}{c_h}$

2. Clearly a linear relationship between x + y, regression line probably doesn't go through the origin, so regression estimator appropriate ✓

$$\hat{\beta} = s_{xy}/s_x^2 = \frac{92.0}{47.7} = 1.93 \quad (3 \text{ sig figs})$$ ✓

$$\hat{\mu}_{lr} = \bar{y} + \hat{\beta}(\mu_x - \bar{x})$$
$$= 215.1 + 1.93(100 - 97.8)$$
$$= 219.3 \quad (3 \text{ sig fig})$$

$$\hat{\tau}_{lr} = 21,930 \quad (\text{—} \; " \; \text{—})$$ ✓

$$\hat{\text{var}}\,\hat{\mu}_{lr} = \frac{1}{n}(1-f)\frac{n-1}{n-2}\left(s_y^2 - s_{xy}^2/s_x^2\right)$$
$$= \frac{1}{10}\cdot\frac{9}{10}\cdot\frac{9}{8}\left(187.2 - (92.0)^2/47.7\right)$$
$$= \frac{81}{800} \times 9.76 = 0.988 \quad (3 \text{ sig fig})$$ ✓✓

95% CI for τ

$$\hat{\tau}_{lr} \pm t_8(.975)\, 100\sqrt{0.988}$$
$$= 21,930 \pm 2.306 \times 99.3$$
$$= 21,930 \pm 229 \quad (3 \text{ sig figs})$$ ✓

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aliter using ratio estimator (max 5/6)

$$\hat{\tau}_{ratio} = 100\,\hat{\mu}_{ratio} = 100\,\frac{\bar{y}}{\bar{x}}\mu_x = 100\,\frac{215.1}{97.8}\,100 = 21,994$$
(5 sig fig) ✓✓

$$\hat{\text{var}}(\hat{\mu}_{ratio}) = \frac{1}{n}(1-f)\left(s_y^2 - 2r\,s_{xy} + r^2 s_x^2\right)$$
$$= \frac{1}{10}\,\frac{9}{10}\left(187.2 - 2\times\frac{215.1}{97.8}\times 92.0 + \left(\frac{215.1}{97.8}\right)^2 \times 47.7\right)$$
$$= 1.193 \quad (4 \text{ sig fig})$$ ✓✓

95% CI for τ

$$\hat{\tau}_{ratio} \pm t_9(.975)\, 100\sqrt{1.193}$$
$$= 21,994 \pm 2.262 \times 109.2$$
$$= 21,994 \pm 247 \quad (3 \text{ sig fig})$$ ✓

max 2/6 for SRS

3. cluster sample: 10 clusters $n = 10, N = 20$

$\text{Var}\,\hat{\mu}_{cl} = S_b^2 (1-f)/n = S_b^2 \cdot \frac{1}{20}$ ✓

simple random sample: sample size 20 $n = 20, N = 100$

$\text{Var}\,\hat{\mu}_{srs} = S^2 (1-f)/n = S^2 \cdot \frac{1}{25}$ ✓

total SS = within SS + between SS

$99\, S^2 = \sum_{i=1}^{20} 4 S_i^2 + \sum_{i=1}^{20} 5 (\mu_i - \mu)^2$

$= 80\, \bar{S}^2 + 95\, S_b^2$ ✓✓

$\text{Var}\,\hat{\mu}_{cl} > \text{Var}\,\hat{\mu}_{srs}$

$\iff S_b^2 \cdot \frac{1}{20} > S^2 \cdot \frac{1}{25}$ ✓

$\iff \frac{99 S^2 - 80 \bar{S}^2}{95} \cdot \frac{1}{20} > S^2 \cdot \frac{1}{25}$ ✓

$\iff 99 \cdot 25\, S^2 - 80 \cdot 25\, \bar{S}^2 > 95 \cdot 20\, S^2$

$\iff 575\, S^2 > 2000\, \bar{S}^2$

$\iff S^2 > 3.48\, \bar{S}^2$ (3 sig figs). ✓✓

prefer cluster sample when $S^2 < 3.48\, \bar{S}^2$

4\. a) $\mathbb{E}\hat{\mu} = \mathbb{E}\,\mathbb{E}(\hat{\mu} \mid \hat{\mu}_1, \ldots, \hat{\mu}_{10})$ ✓

$= \mathbb{E}\,\mathbb{E}(\sum_{i=1}^{5} \hat{\mu}_{k(i)}/5 \mid \hat{\mu}_1, \ldots, \hat{\mu}_{10})$

$= \mathbb{E}\,\mathbb{E}(\sum_{j=1}^{10} I_j \hat{\mu}_j/5 \mid \hat{\mu}_1, \ldots, \hat{\mu}_{10})$ ✓

where $I_j = \begin{cases} 1 & \text{if } j = k(i) \text{ some } i \\ 0 & \text{o/w} \end{cases}$

$= \mathbb{E}\,\frac{1}{10}\sum_{j=1}^{10} \hat{\mu}_j$ since $\mathbb{E} I_j = \frac{5}{10}$ ✓

$= \frac{1}{10}\sum_{j=1}^{10} \mu_j$ since $\mathbb{E}\hat{\mu}_j = \mu_j$ ✓

$= \mu$ (equal sized clusters)

b) original sample:

clusters: $k(1), \ldots, k(5)$

within cluster $k(i)$: $y_{k(i),j}$, $j = 1, \ldots, 10$

bootstrap sample

clusters: $k^*(1), \ldots, k^*(5)$ resampled from $\{k(1), \ldots, k(5)\}$ ✓

for each $k^*(i)$ ✓

resample $y^*_{k(i)^*,j}$ from $\{y_{k^*(i),j}\}_{j=1}^{10}$

bootstrap replicate $\hat{\mu}^* = \sum_{i=1}^{5}(\sum_{j=1}^{10} y^*_{k^*(i),j}/10)/5$

let $\underset{\sim}{y}^*(1), \ldots, \underset{\sim}{y}^*(B)$ be bootstrap samples ✓

and $\hat{\mu}^*(1), \ldots, \hat{\mu}^*(B)$ bootstrap replicates

then
$$\widehat{\mathrm{var}}_B\,\hat{\mu} = \frac{1}{B-1}\sum_{i=1}^{B}(\hat{\mu}^*(i) - \hat{\mu}^*(\cdot))^2$$ ✓

5. i) $(1,0,0)(0,1,1)_{12}$ ✓

ii) $(1-B-B^4+B^5)X_t = (1+\beta_1 B+\beta_4 B^4+\beta_1\beta_4 B^5)Z_t$

$(1-B)(1-B^4)X_t = (1+\beta_1 B)(1+\beta_4 B^4)Z_t$

$(0,1,1)(0,1,1)_4$ ✓✓

iii) $(1-B^{12})^2 X_t = (1-B^{12})^2 Z_t$

$(0,0,0)(0,2,2)_{12}$ ✓✓✓

6. a) MA(1) process $\phi(B)=1$, $\theta(B)=1+\beta B$ ✓✓

$\gamma(0) = \sigma^2(1+\beta^2)$

$\gamma(1) = \sigma^2\beta$ ✓✓ where $\sigma^2 = \mathrm{Var}\, Z_t$

$$\frac{\gamma(1)}{\gamma(0)} = \frac{\beta}{1+\beta^2} = \frac{2}{5}$$

$2\beta^2 - 5\beta + 2 = 0$

$$\beta = \frac{5 \pm \sqrt{25-16}}{4} = 2,\ 1/2$$ ✓

$\beta = 1/2$ gives invertible process ✓✓

b) AR(2) process $\theta(B)=1$, $\phi(B) = 1-\alpha_1 B - \alpha_2 B^2$ ✓✓

auxilliary eqn $\lambda^2 - \alpha_1\lambda - \alpha_2 = (\lambda - 5/6)(\lambda - 1/6)$

$= \lambda^2 - \lambda + 5/36$ ✓✓

so $\alpha_1 = +1$, $\alpha_2 = -5/36$ ✓

Using first two Yule-Walker eqns get $\alpha_1 = \frac{32}{135}$ $\alpha_2 = \frac{7}{135}$

7 $Z_t = X_t - \frac{1}{2} X_{t-1} + Z_{t-1}$

t	X_t	R_t	
0	0	0	
1	8	8	✓
2	-12	-8	✓
3	-6	-8	✓
4	21	16	✓
5	$-5\frac{1}{2}$	0	✓

8 a)

t	1	2	3	4	5	6	7	8	
M_t				1.3	1.4	1.9	2.6	3.1	(1 d.p.)

$\hat{X}_8 = 3.1$ ✓

b)

t	1	2	3	4	5	6	7	8
F_t	1.4	1.4	~~1.87~~ 1.33	1.35 1.31	1.43 1.51	1.87 2.48	2.27 2.98	2.55 3.13
X_t	1.4	1.3	1.3	1.6	2.9	3.2	3.2	

✓✓✓

$\hat{X}_8 = 2.6$ (1 d.p.)

c)

MA SSE for periods 4 – 7

$.3^2 + 1.5^2 + 1.3^2 + .6^2 = 4.4$ (1 d.p.)

Exp Smoother SSE for periods 4 – 7

$.25^2 + 1.47^2 + 1.33^2 + .93^2 = 4.9$ (1 d.p.)

so prefer 3 p.t. moving average smoother.

✓✓✓

9 a) $\hat{F}(x) = \sum_{i=1}^{n} 1\{x \le x_i\}$ ✓

b) Assume $\theta = t(F)$ some ftn t
then $\hat{\theta} = t(\hat{F})$ ✓

c) $\theta = \max_F X = \inf\{x : F(x) = 1\}$ ✓

$\hat{\theta} = \inf\{x : \hat{F}(x) = 1\} = \max_i \{x_i\}$ ✓✓

d) $\hat{F} \to F$ uniformly with prob. 1 as $n \to \infty$
so, if t smooth enough, $t(\hat{F}) \to t(F)$ ✓✓

also $\hat{F}$ is non-parametric ML est of F ✓

$se_{\hat{F}}\,\hat{\theta} = \sqrt{\mathbb{E}_{\hat{F}}\,(S(\underset{\sim}{X}) - \mathbb{E}_{\hat{F}}\,S(\underset{\sim}{X}))^2}$

10 a) ~~let $\underset{\sim}{X}^*(1), \dots, \underset{\sim}{X}^*(B)$ be iid samples fro $\hat{F}$~~
~~and $\hat{\theta}^*(i) = S(\underset{\sim}{X}^*(i))$~~
~~then~~ $\cancel{\widehat{se}_B\,\hat{\theta} = \sqrt{\frac{1}{B-1}\sum(\hat{\theta}^*(i) - \hat{\theta}^*(\cdot))^2}}$

b) i) as per old version of (a) with $B = 250$ ✓✓

ii) $\widehat{se}_B\,\hat{\theta} \approx se_{\hat{F}}\,\hat{\theta} = \mathbb{E}_{\hat{F}}(S(\underset{\sim}{X}) - \mathbb{E}_{\hat{F}}S(\underset{\sim}{X}))^2$

$\approx se_F\,\hat{\theta} = \mathbb{E}_F(S(\underset{\sim}{X}) - \mathbb{E}_F S(\underset{\sim}{X}))^2$

reduce first error by increasing B ✓
reduce second error by increasing n. ✓

iii) max sample ≤ 10 iff all elts of sample ≤ 10
prob. a single elt. ≤ 10 is 0.6
whence
$(0.6)^{10} = 0.006$ (3 d.p.). ✓✓✓

11. a) $\widehat{\text{bias}}_B \hat{\theta} = \hat{\theta}^*(\cdot) - t(\hat{F})$ ✓

$\bar{\theta} = 2\,t(\hat{F}) - \hat{\theta}^*(\cdot)$ ✓

b) $\text{Var}_F \bar{\theta} = 4\,\text{Var}_F\, t(\hat{F}) + \text{Var}_F\, \hat{\theta}^*(\cdot)$

$\approx 4\,(\widehat{se}_B\, \hat{\theta})^2 + \text{Var}_{\hat{F}}\, \hat{\theta}^*(\cdot)$

$\approx (4 + \frac{1}{B})\,(\widehat{se}_B\, \hat{\theta})^2$ ✓✓✓

$\bar{\theta}$ should be unbiased (approx)
so $MSE\ \bar{\theta} = \text{Var}\ \bar{\theta}$ ✓

c) $MSE_F\, \hat{\theta} = \text{Var}_F\, \hat{\theta} + (\text{bias}_F\, \hat{\theta})^2$

$\approx (\widehat{se}_B\, \hat{\theta})^2 + (\widehat{\text{bias}}_B\, \hat{\theta})^2$ ✓

so $MSE_F\, \bar{\theta} - MSE_F\, \hat{\theta}$

$\approx (3 + \frac{1}{B})(\widehat{se}_B\, \hat{\theta})^2 - (\widehat{\text{bias}}_B\, \hat{\theta})^2$

< 0 when $\widehat{\text{bias}}_B\, \hat{\theta} > \sqrt{3 + 1/B}\ \widehat{se}_B\, \hat{\theta}$

$\simeq \sqrt{3}\ \widehat{se}_B\, \hat{\theta}$

for B large. ✓✓✓