620-374 Time Series & Forecasting Exercises

Assignment 4: Exercise 4
Assignment 5: Exercise 8
Assignment 6: Exercises 13 and 14

1. Spencer's 15 point moving average is determined by the weights (to 3 sig. figs.):

-0.009	-0.019	-0.016	0.009	0.066	0.144	0.209	0.231	0.209	0.144	0.066	0.009	-0.016	-0.019	-0.009

Demonstrate that it can be obtained as the composition of the following four moving averages (each is applied in turn).

1/4	1/4	1/4	1/4

1/4	1/4	1/4	1/4

1/5	1/5	1/5	1/5	1/5

-3/4	3/4	1	3/4	-3/4

Hint: express each moving average in terms of a matrix operator.

2. Let $a_1, \ldots, a_m$ be the weights for an m point moving average. Show that the condition

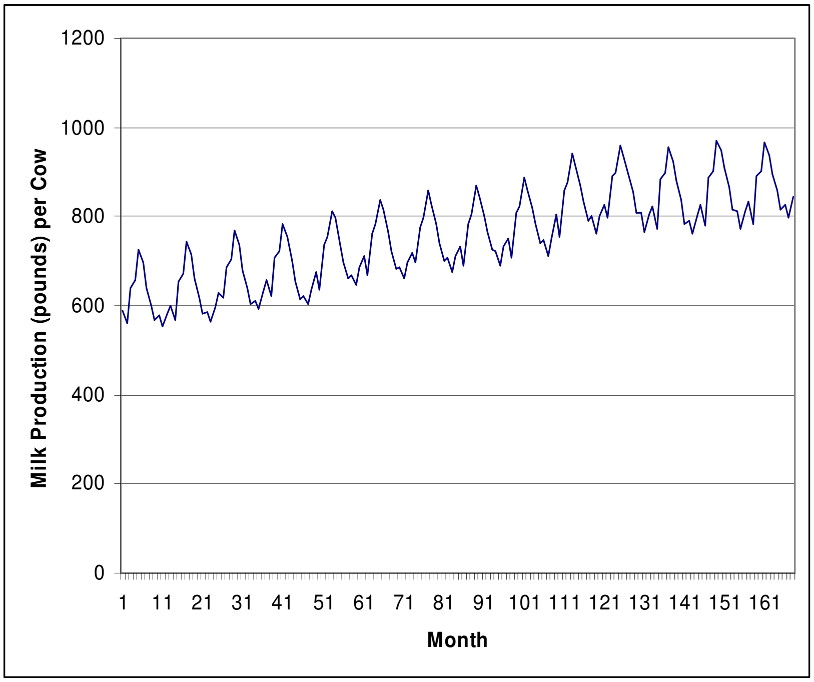
$$\sum_{k = j \bmod s} a_k = \frac{1}{s} \quad \text{for} \quad j = 0, 1, \ldots, s-1$$

is *necessary* for the removal of an additive seasonal effect of period s, in the sense that if it does not hold then we can always find a series for which the seasonal effect will not be removed.

3. Obtain sufficient conditions on the weights used for local regression smoothing to be independent of an additive seasonal effect of period s.

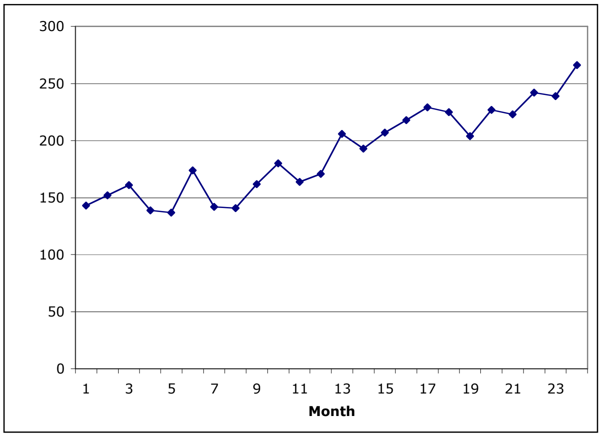
4. The following time-series of monthly milk production per cow (in pounds) can be found at http://www-personal.buseco.monash.edu.au/~hyndman/forecasting/

Divide through by the number of days in a month to get daily milk production, and then decompose the time-series into trend/cycle, seasonal and residual (noise) components, and plot them.



5. The following time-series gives the inventory demand for product E15 over 24 months.

Period	1	2	3	4	5	6	7	8	9	10	11	12
Observation	143	152	161	139	137	174	142	141	162	180	164	171
Period	13	14	15	16	17	18	19	20	21	22	23	24
Observation	206	193	207	218	229	225	204	227	223	242	239	266



Use Holt's method (double exponential smoothing) to forecast period 25. Use values of 0.3 and 0.7 for the smoothing parameters α and β.

For $n = 0, 1, \ldots, 11$ use periods 1 to $12 + n$ to forecast period $13 + n$. Using the Mean Square Error (MSE) find better values of α and β.

6. For the (deterministic) series

$$X_t = t \text{ for } t = 0, 1, \ldots$$

what value of the smoothing parameter α gives the best Exponential Smoothing forecast?

Now suppose that the X_t are independent and identically distributed (i.i.d.) $N(0,1)$ random variables. What are the best lag and α in this case?

Hint: if $X \sim N(0,1)$, then the Minimum Variance Unbiased estimator of X is just 0.

7. Adaptive Response Rate Single Exponential Smoothing is a variant of Exponential Smoothing that continually adjusts the smoothing parameter a to try and allow for changes in the trend.

Let X_t be our time series, F_t the forecast and a_t the smoothing parameter at time t. F_t is updated in the usual way

$$F_{t+1} = a_t X_t + (1 - a_t) F_t$$

The smoothing parameter is defined in terms of the forecasting error $E_t = X_t - F_t$. It is reduced if the forecasting error is "generally" unbiased (mean 0), which would happen if there little trend, and is increased otherwise. For some $0 < b < 1$ we have

$$A_{t+1} = b E_t + (1 - b) A_{t+1}$$
$$M_{t+1} = b |E_t| + (1 - b) M_{t+1}$$
$$a_{t+1} = | A_t/M_t|$$

Apply this to the time series data from Question 5 and compare the results with those given by Holt's method. Use $b = 0.5$

8. There are many variants of the Holt-Winters method, depending upon whether there is no trend, a linear trend, an exponential trend, no seasonality, additive seasonality or multiplicative seasonality. All of them can be written in the following form, with smoothing parameters a, b and c

$$L_t = a P_t + (1 - a) Q_t \qquad \text{(level (seasonally adjusted))}$$
$$B_t = b R_t + (1 - b) B_{t-1} \qquad \text{(slope (seasonally adjusted))}$$
$$\Sigma_t = c T_t + (1 - c) \Sigma_{t-s} \qquad \text{(seasonality of period } s\text{)}$$

Let X_t be our time series, for a linear trend and multiplicative seasonality we have

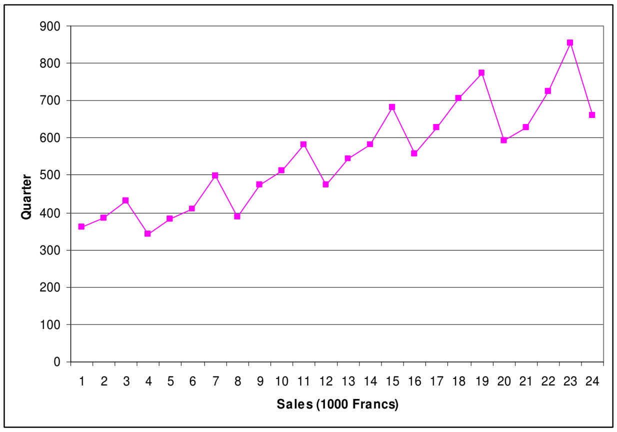
$P_t = X_t/\Sigma_{t-s}$	(seasonally adjusted observation)
$Q_t = L_{t-1} + B_{t-1}$	(projected level (seasonally adjusted))
$R_t = L_t - L_{t-1}$	(observed trend (seasonally adjusted))
$T_t = X_t/L_t$	(observed seasonal effect)

The m step ahead forecast is then

$$F_{t+m} = (L_t + m\,B_t)\,\Sigma_{t+m-s}$$

The following time series gives quarterly sales figures for a French company. It exhibits multiplicative seasonality.

Period	1	2	3	4	5	6	7	8	9	10	11	12
Observation	362	385	432	341	382	409	498	387	473	513	582	474
Period	13	14	15	16	17	18	19	20	21	22	23	24
Observation	544	582	681	557	628	707	773	592	627	725	854	661



Using initial values of 0.3 for each of the smoothing parameters, for $n = 0, 1, \ldots 11$ forecast period $13 + n$ from periods 1 to $12 + n$. By trial and error (or otherwise) find values of a, b and c which reduce the MSE. What is your best forecast for period 25?

9. Suppose that we have a sequence of monthly observations $X_t = (a + bt)S_t + \varepsilon_t$ where S_t has period 12 and ε_t is stationary.

Does the differencing operator ∇_{12} make X_t stationary? Here $\nabla_{12} X_t = X_t - X_{t-12}$. If not find a (combination of) differencing operators which does.

10. Let $\{Z_t\}_{t\geq 0}$ be a sequence of i.i.d. r.v.s with mean 0 and variance σ^2. Put

$$X_t = Z_t + \theta Z_{t-1}$$
$$Y_t = Z_t + \frac{1}{\theta} Z_{t-1}$$

Show that X_t and Y_t have the same autocorrelation function.

By rearranging the equations above, write Z_t as a linear combination of $X_t, X_{t-1}, \ldots$ and as a linear combination of $Y_t, Y_{t-1}, \ldots$. Show that in at most one case the coefficients are absolutely summable. (In this case we say that the model is *invertible*. It can be shown that there is at most one invertible model for any given autocorrelation function.)

11. Only answer this question if you know what a Markov chain is. Let *X* be a positive recurrent aperiodic discrete-time chain, with equilibrium distribution π. Suppose that *X* is in equilibrium, that is $\mathrm{P}(X_t = x) = \pi(x)$. Show that *X* is stationary.

Note that Markov models and ARMA models do not overlap much.

For Questions 12 to 16 let $\{Z_t\}_{t=0}^{\infty}$ be an i.i.d. sequence with mean μ and variance σ^2.

12. Let $\{X_t\}_{t=0}^{\infty}$ be the moving average (MA) process given by

$$X_t = \sum_{k=0}^{m} \frac{1}{m+1} Z_{t-k}$$

Show that the autocorrelation function of this process is

$$\rho(k) = \begin{cases} (m+1-k)/(m+1) & \text{for } k = 0,1,\ldots,m \\ 0 & \text{for } k > m \end{cases}$$

13. Find those values of $a, b \geq 0$ that make the following autoregressive (AR) process stationary

$$X_t = aX_{t-1} + bX_{t-2} + Z_t.$$

For *a* = 1/3 and *b* = 2/9 show that $\{X_t\}_{t=0}^{\infty}$ has autocorrelation function

$$\rho(k) = \frac{16}{21}\left(\frac{2}{3}\right)^k + \frac{5}{21}\left(-\frac{1}{3}\right)^k \text{ for } k = 0,1,2,\ldots$$

14. For each of the following models express the model in terms of the shift operator B acting on X_t and Z_t and then determine whether the model is stationary and/or invertible.

(a) $X_t = 0.3X_{t-1} + Z_t$

(b) $X_t = Z_t - 1.3Z_{t-1} + 0.4Z_{t-2}$

(c) $X_t = 0.5X_{t-1} + Z_t - 1.3Z_{t-1} + 0.4Z_{t-2}$

What is the MA representation of model (a)?

15. Suppose that $\{X_t\}_{t=0}^{\infty}$ is a stationary process and can be written as either $X_t = \psi(B)Z_t$ or $\pi(B)X_t = Z_t$, where B is the shift operator. We define the autocovariance generating function by

$$\Gamma(s) = \sum_{k=-\infty}^{\infty} \gamma(k)s^k$$

where $\gamma(k) = \text{Cov}(X_t, X_{t+k})$. By equating coefficients of s^k show that

$$\Gamma(s) = \sigma^2\psi(s)\psi(1/s) = \frac{\sigma^2}{\pi(s)\pi(1/s)}.$$

16. Find the partial autocorrelation of the AR(2) process $X_t = \frac{1}{3}X_{t-1} + \frac{2}{9}X_{t-2} + Z_t$ (see Question 13).

17. The following is a realisation of a sequence of i.i.d. N(0,1) random variables, $\{Z_t\}_{t=1}^{30}$.

0.01	1.38	0.53	1.58	1.32	1.04	0.33	-0.20	1.90	0.72
-0.27	-1.43	-1.15	-0.07	1.69	0.28	0.01	0.94	-2.10	0.09
0.91	1.76	0.84	-1.13	0.92	1.67	-1.03	-1.71	1.18	-0.59

Use them to generate realisations of the following sequences. In each case plot the series and (using suitable software) plot the sample autocorrelation and sample partial autocorrelation.

(a) The AR(1) sequence $X_t = 0.6X_{t-1} + Z_t$, with $X_0 = 0$.

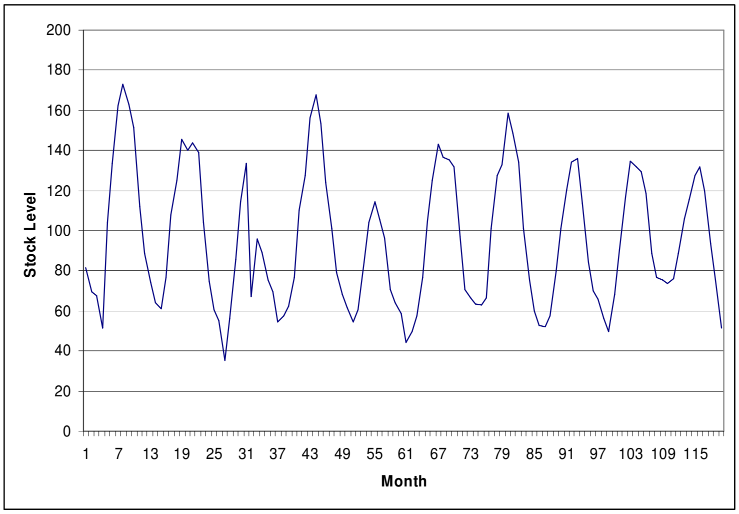
(b) The MA(1) sequence $X_t = Z_t + 0.6Z_{t-1}$, with $Z_0 = 0$.

(c) The AR(2) sequence $X_t = -0.8X_{t-1} + 0.3X_{t-2} + Z_t$, with $X_0 = X_{-1} = 0$.

(d) The MA(2) sequence $X_t = Z_t + 0.8Z_{t-1} - 0.3Z_{t-2}$, with $Z_0 = Z_{-1} = 0$.

(e) The ARMA(1,1) sequence $X_t = 0.6X_{t-1} + Z_t + 0.6Z_{t-1}$, with $X_0 = 0$ and $Z_0 = 0$.

18. The following time-series gives a particular manufacturer's stock level of Evaporated and Sweet Condensed Milk over the period Jan 1971 to Dec 1980.



The original data can be found at
http://www-personal.buseco.monash.edu.au/~hyndman/forecasting/

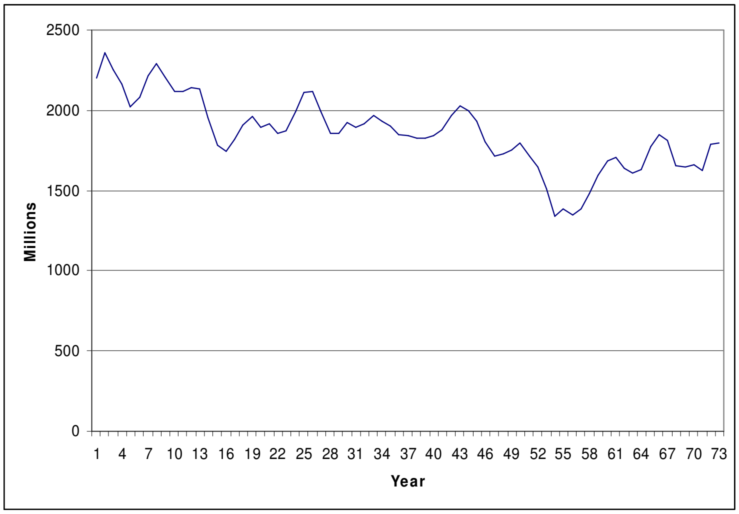
Calculate the autocorrelation function (acf) and partial autocorrelation function (pacf) for this series.

The data clearly has (additive) seasonality with period 12. Apply a lag 12 differencing to remove the seasonality and recalculate the acf and pacf.

There is some indication of a (local) trend. Apply a further lag 1 differencing to remove any trend and recalculate the acf and pacf. Does the series look stationary now? Suggest a model for the differenced series.

Express your model (including differencing) using the shift operator *B*.

19. The following time series gives the sheep population in England and Wales from 1867 to 1939.



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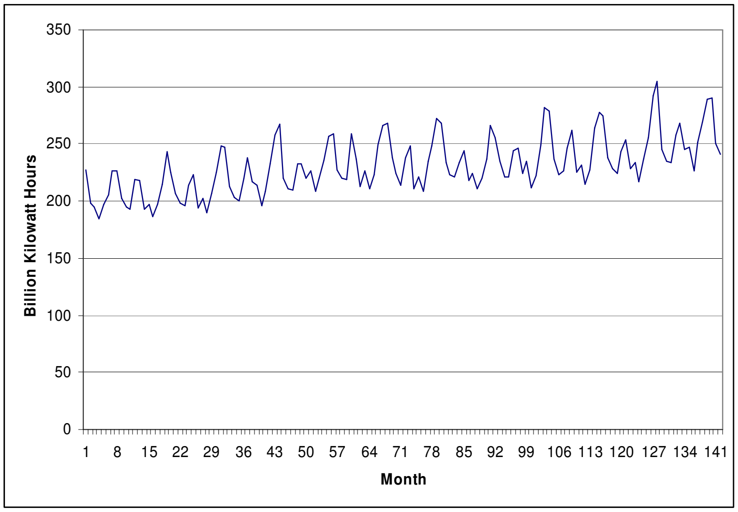
There appears to be a linear trend. Accordingly apply a lag 1 differencing then calculate the acf and pacf.

Fit an AR(p) model to the differenced data. Use it to forecast ahead a further 3 years.

20. The following time series gives the net generation of electricity in the U.S.A. for the period Jan 1985 to Oct 1996.



The original data can be found at http://www-personal.buseco.monash.edu.au/~hyndman/forecasting/

Fit an ARIMA model to this data and use it to forecast ahead 24 months.

Check that the residuals resemble white noise by calculating their acf.

The seasonality looks like it could be multiplicative. Accordingly take logs of the original data and repeat your analysis. How do your new forecasts compare to the old ones?

If you wish to check your forecasts against the real thing, more recent data is available from http://www.eia.doe.gov/emeu/mer/elect.html