Weight Distribution of the Bases of a Binary Matroid

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Abstract—Let $M$ be a weighted binary matroid and $w_1 < \cdots < w_m$ be the increasing sequence of all possible distinct weights of bases of $M$. We give a sufficient condition for the property that $w_1, \ldots, w_m$ is an arithmetical progression of common difference $d$. We also give conditions which guarantee that $w_{i+k} - w_i \leq d, 1 \leq i \leq m - 1$. Dual forms for these results are given also. © 1998 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Let $G = (V(G), E(G))$ be a connected graph and $\mathcal{F}(G)$ the set of spanning trees of $G$. Let $w : E(G) \rightarrow \mathbb{R}$ be a weight function which associates a real number weight $w(e)$ with each edge $e \in E(G)$. For each $T \in \mathcal{F}(G)$, the weight of $T$ is $w(T) = \sum_{e \in E(T)} w(e)$. Denote all distinct weights of spanning trees of $G$ by $w_1 > \cdots > w_m$. The spanning trees with weight $w_i$ are called the $i$th maximal spanning trees. For each $T \in \mathcal{F}(G)$ and integer $k, 0 \leq k \leq |V(G)|$, let $L_k(T) = \{ T' \in \mathcal{F}(G) : |T' \setminus T| \leq k \}$. Kano [1], conjectured that for any maximum weight spanning tree $A$ and each $i$ with $1 \leq i \leq k$, $L_k-1(A)$ contains an $i$th maximal spanning tree of $G$. He proved [1] that the conjecture is true when $w_1, \ldots, w_m$ is an arithmetical progression. Although the conjecture has been fully proved [2,3], we feel that the problem of when $w_1, \ldots, w_m$ is an arithmetical progression is of interest for its own reason. In this direction, an early result of Hakimi and Maeda [4] says that if the weight $w(e)$ of each edge $e$ is $c, c+d$, or $c+2d$ for some constants $c$ and $d > 0$, then $w_1, \ldots, w_m$ is an arithmetical progression. On the other hand, it seems that we do not know much about the distribution of the weights of spanning trees of a graph, although a lot of combinatorial optimization problems, such as the minimum spanning tree problem, relate closely to the weights of spanning trees. In general, it is difficult to have a detailed understanding of the distribution of the weights of bases of a weighted matroid.

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In this paper, we tentatively give a condition which guarantees that the weights of bases of a weighted binary matroid consist of an arithmetical progression. Also we give a sufficient condition for the property that for each \( i \), the difference of the \((i+1)\)th minimal and the \(i\)th minimal weights does not exceed a constant \(d\). The dual versions of these results are provided.

2. MAIN RESULTS AND THE PROOF

The reader is referred to [5] for terminologies on matroids. Let \( M \) be a matroid on a finite set \( S \) and \( B(M) \) the set of bases of \( M \). For any \( B \in B(M) \) and \( x \in S \setminus B \), \( B \cup \{x\} \) contains a unique circuit \( C(x,B) \), called the fundamental circuit of \( x \) in the base \( B \). Note that \( x \in C(x,B) \).

**Lemma 1.** (See [5].) Suppose \( B \in B(M), x \in S \setminus B, y \in B \). Then \( (B \setminus \{y\}) \cup \{x\} \in B(M) \) if and only if \( y \in C(x,B) \) or \( y = x \).

If for any two distinct circuits \( C_1, C_2 \) of \( M \), the symmetric difference \( C_1 \Delta C_2 \) contains a circuit, then \( M \) is said to be a binary matroid [5]. Note that there are alternative ways to define a binary matroid. We present the following equivalent condition which will be used later.

**Lemma 2.** (See [5].) \( M \) is a binary matroid if and only if the symmetric difference of any collection of distinct circuits is the union of disjoint circuits of \( M \).

In the following, we always suppose \( M \) is a binary matroid on finite \( S \). For a subset \( X \) of \( S \), the incidence vector of \( X \) is the vector \( (i_x)_{x \in X} \) with entries indexed by the elements of \( S \), where \( i_x \) is 1 or 0 depending on whether \( x \) is or is not in \( X \). The circuit space of \( M \), denoted by \( V(M) \), is the vector space over the field \( GF(2) \) generated by the incidence vectors of the circuits of \( M \). We can view the vectors of \( V(M) \) as symmetric differences of some circuits of \( M \) (or equivalently as disjoint union of some circuits). The sum of \( X, Y \in V(M) \) is the symmetric difference \( X \Delta Y \). We call a base of \( V(M) \) a circuit base if each vector in this base is a circuit of \( M \). Note that the dimension of \( V(M) \) is \( \rho = |S| - r \), where \( r = \text{rank}(M) \) is the rank of \( M \).

Let \( w : S \rightarrow \mathbb{R} \) be a weight function, where \( \mathbb{R} \) is the set of real numbers. Thus, \( M \) is a weighted matroid with weight \( w(x) \) for each \( x \in S \). The weight of a base \( B \in B(M) \) is \( w(B) = \sum_{x \in B} w(x) \). A base with maximum weight is said to be a maximum base. Suppose \( w_1 < \cdots < w_m \) is the sequence of all distinct weights of bases of \( M \). In this section, we always suppose that the following condition is satisfied.

**Condition.** There exists a circuit base \( \mathcal{C} = \{C_1, \ldots, C_{\rho}\} \) of \( V(M) \) such that for each \( C_i \) there exists at most one \( C_j \) with \( C_i \cap C_j \neq \emptyset, j \neq i \).

For the case of a cycle matroid of a graph \( G \), this condition is satisfied when, for example, the cycles of \( G \) are pairwise edge disjoint. We have the following.

**Lemma 3.**

(i) If \( x_1, \ldots, x_\rho \in S \) satisfy

\[
x_i \in C_i \setminus \bigcup_{j \neq i} C_j, \quad 1 \leq i \leq \rho,
\]

then \( B = S \setminus \{x_1, \ldots, x_\rho\} \in B(M) \) and \( C_i = C(x_i,B) \).

(ii) Conversely, for any \( B \in B(M) \) there exists an order \( x_1, \ldots, x_\rho \) of the elements of \( S \setminus B \) such that (1) is satisfied.

**Proof.**

(i) Since \( |B| = |S \setminus \{x_1, \ldots, x_\rho\}| = r \), it suffices to show that \( B \) is an independent set. Suppose otherwise, then there exists a circuit \( C \) which is contained in \( B \). Since \( C \) is a base for the vector space \( V(M) \), \( C \) can be expressed as \( C_{i_1} \Delta \cdots \Delta C_{i_k}, 1 \leq i_1 < \cdots < i_k \leq \rho \). From (1) we have \( x_{i_1} \in C \subseteq B \), a contradiction. So \( B \) is an independent set and hence \( B \in B(M) \). By \( C_i \setminus \{x_i\} \subseteq B \), we know \( C_i = C(x_i,B) \).
(ii) We need to prove that there exists a bijection $f : S \setminus B \to C$ such that $x \in f(x)$, $x \notin f(y)$ for any distinct $x, y \in S \setminus B$.

For any $x \in S \setminus B$, let $C(x, B) = C_{i_1} \triangle \cdots \triangle C_{i_k}, 1 \leq i_1 < \cdots < i_k \leq \rho$. From the above-mentioned condition and $x \in C(x, B)$, we know $x$ belongs to exactly one $C_{i_k}$. Without loss of generality, suppose $x \in C_{i_1} \setminus \bigcup_{i=2}^{k} C_{i_i}$. Set $f(x) = C_{i_1}$. In this way, we define a mapping $f$ from $S \setminus B$ to $C$. For $y \in S \setminus B, y \neq x$, let $C(y, B) = C_{j_1} \triangle \cdots \triangle C_{j_l}, 1 \leq j_1 < \cdots < j_l \leq \rho$. Also, we may suppose $y \in C_{j_1} \setminus \bigcup_{i=2}^{l} C_{j_i}$. Then $f(y) = C_{j_1}$. Now we prove

$$f(x) \neq f(y)$$

and

$$x \notin f(y).$$

If these are achieved, then from (2), we know $f$ is injective and hence bijective since $|S \setminus B| = |C|$, and from (3), we get (1).

Let us prove (2) first. Suppose to the contrary that $f(x) = f(y)$, i.e., $C_{i_1} = C_{j_1}$. Then $k, l \geq 2$. In fact, if $k = 1$, then from $y \in C_{j_1}$, $C_{i_1} = C(x, B)$ we know $y \in C(x, B) \setminus \{x\} \subset B$, a contradiction. Similarly, $l \geq 2$. Since $y \notin C(x, B)$ but $y \in C_{i_1}$, there exists, say, $C_{i_2}$ which contains $y$. From the above-mentioned condition, we have $y \notin \bigcup_{i=3}^{k} C_{i_i}$. Similarly, we can suppose $x \in C_{j_2}$ and $x \notin \bigcup_{i=3}^{l} C_{j_i}$. Note that $C_{i_1} \neq C_{i_2}, C_{j_2}$, but $C_{i_1} \cap C_{i_2} \neq \emptyset, C_{i_1} \cap C_{j_2} \neq \emptyset$. This contradicts the hypothesis of the condition, and hence, (2) follows.

Now, we prove (3). If $x \in f(y) = C_{j_1}$, then there exists exactly one $C_{i_1}$ such that $x \in C_{i_1}, t \geq 2$. Without loss of generality, we suppose $x \in C_{j_2}$. Then we must have $C_{i_1} = C_{j_2}$, since otherwise, the pairwise distinct $C_{i_1}, C_{j_1}, C_{j_2}$ will have a common element $x$, violating the hypothesis in the condition. We claim that there exist no $x$ with $x \in C_{j_1} \setminus B, x \neq x, y$. Suppose otherwise, then by $y \notin C(x, B)$ and by the condition, we know there exists a unique $C_{j_2}$ with $x \in C_{j_2}, t \geq 2$. If $t > 2$, then $C_{i_1}$ has nonempty intersection with both $C_{j_2}$ and $C_{j_1}$, a contradiction. So we have $t = 2$. That is, $x \in C_{j_1} = C_{i_1}$. But $x \notin C(x, B)$, so there exists a unique $C_{i_2}$ with $x \in C_{i_2}, s \geq 2$. Note that $C_{i_2} \neq C_{j_1}$, for otherwise $x$ will be in $C_{j_1}$. Thus, $C_{j_1}$ has nonempty intersection with $C_{i_1}$ and $C_{i_2}$, which contradicts the condition. So there exist no $x$ with $x \in C_{j_1} \setminus B, x \neq x, y$, and hence, $C(x, B) \Delta C(y, B) \Delta C_{j_1} \subset B$. But $M$ is binary implies that $C(x, B) \Delta C(y, B) \Delta C_{j_1}$ is the union of disjoint circuits. So the base $B$ must contain circuits. This contradiction completes the proof of (3) and hence of Lemma 3.

**Lemma 4.** Suppose $B \in B(M)$ and $S \setminus B = \{x_1, \ldots, x_p\}$ satisfies (1). Then $B$ is a maximum base if and only if for each $i$ is a minimum weight element in $C_{i_1}, 1 \leq i \leq \rho$.

**Proof.** Suppose $x_i$ is not a minimum weight element of $C_{i_1}$ for some $i$. Then there exists $y_i \in C_{i_1} \setminus \{x_i\}$ with $w(y_i) < w(x_i)$. By Lemma 3, we have $C_{i_1} = C(x_i, B)$, and hence, $(B \setminus \{y_i\}) \cup \{x_i\} \in B(M)$. $B$ is not a maximum base since $w((B \setminus \{y_i\}) \cup \{x_i\}) = w(B) - w(y_i) + w(x_i) > w(B)$.

Conversely suppose each $x_i$ is a minimum weight element in $C_{i_1}$. By Lemma 3, for any $B' \in B(M)$, the elements of $S \setminus B'$ can be ordered as $x_1', \ldots, x_p'$ such that $x_i \in C_{i_1} \setminus \bigcup_{j \neq i} C_{j_1}$. Since $w(x_i) \leq w(x_i'), 1 \leq i \leq \rho$, we have $w(B') = w(B) + \sum_{i=1}^{p} (w(x_i) - w(x_i')) \leq w(B)$, and hence, $B$ is a maximum base. This completes the proof of Lemma 4.

For a circuit $C$ of $M$, let $c_1 < \cdots < c_n$ be all distinct weights of elements of $C$. If $c_1, \ldots, c_n$ is an arithmetical progression with common difference $d$, for some real number $d > 0$, then $C$ is said to satisfy the $d$-condition. If $c_{i+1} - c_i \leq d, 1 \leq i \leq n - 1$, then we say $C$ satisfies the $d^*$-condition. We have the following lemma.

**Lemma 5.** Suppose $B \in B(M)$ is not a maximum base. Then

(i) if each $C_{i_1}$ satisfies the $d$-condition, $1 \leq i \leq \rho$, then there exists $B' \in B(M)$ such that $w(B') = w(B) + d$;

(ii) if each $C_{i_1}$ satisfies the $d^*$-condition, $1 \leq i \leq \rho$, then there exists $B' \in B(M)$ such that $w(B) < w(B') \leq w(B) + d$. 
Proof. By Lemma 3, we can suppose $S \setminus B = \{x_1, \ldots, x_\rho\}$ satisfies (1) and $C_i = C(x_i, B), 1 \leq i \leq \rho$. If each $C_i$ satisfies the $d$-condition, then by Lemma 4 and the assumption that $B$ is not a maximum base, we know there exist $C_i$ and $x'_i \in C_i \setminus \{x_i\}$ such that $w(x'_i) = w(x_i) - d$. By Lemma 1, $B' = (B \setminus \{x'_i\}) \cup \{x_i\} \in B(M)$. The weight of $B'$ is $w(B') = w(B) - w(x'_i) + w(x_i) = w(B) + d$. In a similar way, one can prove (ii).

From Lemma 5, we get our main result.

Theorem 1. Suppose $S, M, w, w_i$ are as before and $d$ is a positive number. Suppose there exists a circuit base $C = \{C_1, \ldots, C_\rho\}$ of $V(M)$ which satisfies the condition.

(i) If each $C_i$ satisfies the $d$-condition, then $w_1, \ldots, w_m$ is an arithmetical progression with common difference $d$.

(ii) If each $C_i$ satisfies the $d^2$-condition, then $0 < w_{i+1} - w_i \leq d$, $1 \leq i \leq m - 1$.

An integer interval is a set of consecutive integers. From Theorem 1, we have the following.

Corollary 1. Suppose $M$ is a binary matroid on $S$ and there exists a circuit base $C$ of $V(M)$ which satisfies the condition. If $w$ is an integer-valued weight function defined on $S$ such that the weights of the elements in each $C_i$ consist of an integer interval, then the weights of the bases of $M$ also consist of an integer interval.

3. DUAL THEOREM

The cocircuit space $V^*(M)$ of $M$ is the vector space over $GF(2)$ generated by the incidence vectors of the cocircuits of $M$. The dimension of $V^*(M)$ is $r$. A base $C^*_1, \ldots, C^*_r$ of $V^*(M)$ is said to be a cocircuit base if each $C^*_i$ is a cocircuit of $M$. Let $B^*(M)$ be the set of cobases of $M$. The weight of a cobase $B^*$ is $w(B^*) = \sum_{x \in B^*} w(x)$. Let $w^*_1 < \cdots < w^*_m$ be all the possible distinct weights of cobases of $M$. From Theorem 1 and the duality principle [5] for matroids, we get the following.

Theorem 2. Suppose $S, M, w, w^*_i$ are as before and $d$ is a positive number. Suppose there exists a cocircuit base $C^* = \{C^*_1, \ldots, C^*_r\}$ of $V^*(M)$ such that each $C^*_i$ has nonempty intersection with at most one $C^*_j, j \neq i$.

(i) If each $C^*_i$ satisfies the $d$-condition, then $w^*_1, \ldots, w^*_m$ is an arithmetical progression with common difference $d$.

(ii) If each $C^*_i$ satisfies the $d^2$-condition, then $0 < w^*_{i+1} - w^*_i \leq d$, $1 \leq i \leq m - 1$.

Corollary 2. Suppose $M$ is a binary matroid on $S$ and there exists a cocircuit base $C^* = \{C^*_1, \ldots, C^*_r\}$ of $V^*(M)$ such that each $C^*_i$ intersects at most one other $C^*_j$. If $w$ is an integer-valued weight function for $M$ such that the weights of the elements in each $C^*_i$ consist of an integer interval, then the weights of the cobases of $M$ also consist of an integer interval.

In particular, the corollaries of Theorems 1 and 2 are valid for the cycle and cocycle matroids of a graph since they are both binary.

References