SOLUTION TO A QUESTION ON A FAMILY OF IMPRIMITIVE SYMMETRIC GRAPHS

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Abstract

We answer a recent question posed by Li et al. [‘Imprimitive symmetric graphs with cyclic blocks’, European J. Combin. 31 (2010), 362–367] regarding a family of imprimitive symmetric graphs.

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A graph $G = (V, E)$ is called $G$-symmetric if $G$ admits $G$ as a group of automorphisms such that $G$ is transitive on $V$ and on the set of arcs of $G$, where an arc is an ordered pair of adjacent vertices. If in addition $G$ admits a nontrivial $G$-invariant partition, that is, a partition $B$ of $V$ such that $1 < |B| < |V|$ and $B^g := \{\alpha^g : \alpha \in B\} \subseteq B$ for $B \in B$ and $g \in G$, then $G$ is called an imprimitive $G$-symmetric graph. In this case the quotient graph $G_B$ of $G$ with respect to $B$ is defined to have vertex set $B$ such that $B, C \in B$ are adjacent if and only if there exists at least one edge of $G$ between $B$ and $C$. We assume that $G_B$ contains at least one edge, so that each block of $B$ is an independent set of $G$. Denote by $\Gamma(\alpha)$ the neighbourhood of $\alpha \in V$ in $G$, and define $\Gamma(X) = \bigcup_{\alpha \in X} \Gamma(\alpha)$ for $X \in B$. For blocks $B, C \in B$ adjacent in $G_B$, let $\Gamma[B, C]$ be the bipartite subgraph of $\Gamma$ induced on $(B \cap \Gamma(C)) \cup (C \cap \Gamma(B))$. Then $\Gamma[B, C]$ is independent of the choice of $(B, C)$ up to isomorphism. Define

$$v := |B| \quad \text{and} \quad k := |B \cap \Gamma(C)|$$

to be the block size of $B$ and the size of each part of the bipartition of $\Gamma[B, C]$, respectively.

In line with a geometrical approach suggested in [1], various situations may occur for $\Gamma, G, \Gamma_B, \Gamma[B, C]$ and a certain 1-design with point set $B$; see, for example, [1, 3, 5–7]. The case where $k = v - 2 \geq 1$ was studied in [2, 4] and a necessary and sufficient condition for $\Gamma_B$ to be $(G, 2)$-arc-transitive was given in [2]. In this case, the multigraph $\Gamma^B$ [2] with vertex $B$ and an edge joining the two vertices of $B \setminus \Gamma(C)$ for every $C \in \Gamma_B(B)$ plays an important role in the structure of $\Gamma$ and $\Gamma_B$, respectively.

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where $\Gamma_B(B)$ is the neighbourhood of $B$ in $\Gamma_B$. Since $\Gamma$ is $G$-symmetric, up to isomorphism $\Gamma^B_B$ is independent of the choice of $B$, and the multiplicity of each edge $\{\alpha, \beta\}$ of $\Gamma^B_B$, namely
\[
m := |\{C \in \Gamma_B(B) : B \setminus \Gamma(C) = \{\alpha, \beta\}\}|,\]
is independent of the choice of $\{\alpha, \beta\}$. Denote by $\text{Simple}(\Gamma^B_B)$ the underlying simple graph of $\Gamma^B_B$ and by $G_B$ the setwise stabilizer of $B$ in $G$. It has been proved [2, Theorem 2.1] that $\text{Simple}(\Gamma^B_B)$ is $G_B$-vertex-transitive and $G_B$-edge-transitive, and either $\Gamma^B_B$ is connected or $v$ is even and $\text{Simple}(\Gamma^B_B)$ is a perfect matching $(v/2) \cdot K_2$.

In the latter case detailed information about $\Gamma$ was obtained in [2, Theorem 1.3] when $\Gamma^B_B$ is simple. In [4], Li et al. proved that, if $\text{Simple}(\Gamma^B_B)$ is a cycle, then $v$ must be small, namely $v$ is equal to 3 or 4. Based on this they posed the following question.

**Question 1.** In the case where $k = v - 2$ and $\Gamma^B_B$ is connected, is $v$ bounded by some function of the valency of $\text{Simple}(\Gamma^B_B)$?

Define
\[
b := \text{val}(\Gamma_B), \quad s := \text{val}(\Gamma[B, C]), \quad r := |\{C \in B : \alpha \in \Gamma(C)\}|\]
to be respectively the valency of $\Gamma_B$, the valency of $\Gamma[B, C]$, and the number of blocks of $B$ that contain at least one neighbour of a fixed vertex $\alpha \in V$ in $\Gamma$. Note that $v, k, b, r$ and $s$ all rely on the $G$-invariant partition $B$.

In this paper we answer Question 1 by proving the following stronger result: there are only two possibilities for $\text{Simple}(\Gamma^B_B)$ and $v$ can take two values only.

**Theorem 2.** Suppose that $\Gamma$ is a $G$-symmetric graph which admits a nontrivial $G$-invariant partition $B$ such that $k = v - 2 \geq 1$, $\Gamma_B$ is connected and $\text{Simple}(\Gamma^B_B)$ is connected with valency $d \geq 2$. Then one of the following occurs.

(a) $\text{Simple}(\Gamma^B_B) \cong K_v, \; v = d + 1, \; b = m(v - 1)v/2$, and $G_B$ is 2-homogeneous.

(b) $\text{Simple}(\Gamma^B_B) \cong K_{v/2,v/2}, \; v = 2d, \; b = mv^2/4$, and the bipartition of $\text{Simple}(\Gamma^B_B)$ induces a $G$-invariant partition $B^*$ of the vertex set of $\Gamma$ (which is a refinement of $B$) such that one of the following holds for its parameters:

(i) $v^* = k^* + 1 = v/2, \; b^* = b, \; r^* = r, \; s^* = s$;

(ii) $v^* = k^* + 1 = v/2, \; b^* = 2b, \; r^* = 2r, \; s^* = s/2$;

(iii) $v^* = 2k^* + 1 = v/2, \; b^* = 2b, \; r^* = r, \; s^* = s$.

**Proof.** Suppose that $\Gamma$, $G$ and $B$ satisfy the conditions in the theorem. Denote $\Omega := \text{Simple}(\Gamma^B_B)$. Let $B$ and $C$ be two blocks of $B$ adjacent in $\Gamma_B$, and let $\{\alpha, \beta\} = B \setminus \Gamma(C)$ be the corresponding edge of $\Omega$. Define
\[
U := (\Omega(\alpha) \cup \Omega(\beta)) \setminus \{\alpha, \beta\}\]
to be the neighbourhood of the subset $\{\alpha, \beta\}$ of $B$ in $\Omega$, and set
\[
W := B \setminus (U \cup \{\alpha, \beta\})\].
Since \( \Omega \) has valency \( d \geq 2 \), we have \( U \neq \emptyset \). Since every element of \( G_{BC} = (G_B)_C \) fixes \( \{ \alpha, \beta \} \) setwise, it follows that every element of \( G_{BC} \) fixes each of \( U \) and \( W \) setwise. Thus \( G_{BC} \leq G_U \cap G_W \).

Claim 1. \( W = \emptyset \), that is, \( U = B \setminus \{ \alpha, \beta \} \), or every vertex in \( B \) is adjacent to at least one of \( \alpha \) and \( \beta \) in \( \Omega \).

Suppose otherwise and let \( \delta \in W \). Since \( U \neq \emptyset \), we may take a vertex \( \gamma \in U \). Since \( \delta, \gamma \not= \alpha, \beta \), there exist \( \delta_1, \gamma_1 \in C \) adjacent to \( \delta, \gamma \) in \( \Gamma \), respectively. (It may occur that \( \delta_1 = \gamma_1 \).) Since \( \Gamma \) is \( G \)-symmetric, there exists \( g \in G \) such that \( (\gamma, \gamma_1)^g = (\delta, \delta_1) \). Since \( g \) maps \( \gamma \in B \) to \( \delta \in B \) and \( \gamma_1 \in C \) to \( \delta_1 \in C \), it fixes \( B \) and \( C \) setwise. Hence \( g \in G_{BC} \leq G_U \cap G_W \). However, this is a contradiction, because \( g \) maps \( \gamma \in U \) to \( \delta \in W \). Therefore \( W = \emptyset \) as claimed.

Since \( \Omega \) has valency \( d \), by Claim 1, \( d - 1 \leq |U| \leq 2(d - 1) \). Since \( v = |U| + 2 \) by Claim 1, it follows that
\[
d + 1 \leq v \leq 2d.
\]

Claim 2. In \( \Omega \) any two adjacent vertices have \( 2d - v \) common neighbours, and two nonadjacent vertices have the same neighbourhood.

In fact, since \( \Omega \) is \( G_B \)-edge-transitive [2, Theorem 2.1], the number \( \lambda \) of common neighbours of a pair of adjacent vertices in \( \Omega \) is a constant. Consider the neighbourhood \( U \) of \( \{ \alpha, \beta \} \) in \( \Omega \), where \( \alpha \) and \( \beta \) are as above. There are exactly \( d - \lambda - 1 \) vertices in \( B \) which are adjacent to \( \alpha \) but not \( \beta \) (\( \beta \) but not \( \alpha \), respectively). Thus, by Claim 1, \( 2(d - \lambda - 1) + \lambda = v - 2 \), which implies that \( \lambda = 2d - v \).

Now let \( \sigma \) and \( \tau \) be any two nonadjacent vertices of \( \Omega \). If \( \gamma \in B \) is adjacent to \( \sigma \) in \( \Omega \), then by applying Claim 1 to the edge \( \{ \sigma, \gamma \} \), every vertex in \( B \) is adjacent to either \( \sigma \) or \( \gamma \) in \( \Omega \). Thus, since \( \tau \) is not adjacent to \( \sigma \), it must be adjacent to \( \gamma \) in \( \Omega \) and so \( \Omega(\sigma) \subseteq \Omega(\tau) \). Similarly, \( \Omega(\tau) \subseteq \Omega(\sigma) \). Hence \( \Omega(\sigma) = \Omega(\tau) \) and Claim 2 is proved.

Consider any maximal (with respect to set-theoretic inclusion) independent set \( X \) of \( \Omega \). By Claim 2 the vertices in \( X \) have the same neighbourhood in \( \Omega \). Denote this common neighbourhood by \( Y \), so that \( |Y| = d \). If \( B \setminus (X \cup Y) \neq \emptyset \), then by the maximality of \( X \), any vertex in \( B \setminus (X \cup Y) \) must be adjacent to at least one vertex \( \delta \in X \) in \( \Omega \), which implies that \( \delta \) is adjacent to \( d + 1 \) vertices in \( \Omega \). This contradiction shows that \( X \cup Y = B \) and consequently \( |X| = v - d \). Since this holds for any maximal independent set of \( \Omega \) and since \( \Omega \) is \( G_B \)-vertex-transitive, we have the following claim.

Claim 3. \( v - d \) divides \( d \) and \( \Omega \) is a complete \( t \)-partite graph with each part containing \( v - d \) vertices, where \( t = v/(v - d) \).

Based on this we now prove the following claim.

Claim 4. \( \Omega \cong K_v \) or \( K_{v/2,v/2} \); that is, \( t = v \) or \( 2 \).
Suppose to the contrary that $2 < t < v$. Denote by $B_1, B_2, \ldots, B^t$ the parts of the $t$-partition of $\Omega$. Similarly, for any $D \in B$, denote by $D_1, D_2, \ldots, D^t$ the parts of the $t$-partition of $\text{Simple}(\Gamma^D) (\cong \Omega)$. Set

$$B^* := \{D_1, D_2, \ldots, D^t : D \in B\}.$$  

It is straightforward to verify that $B^*$ is a nontrivial $G$-invariant partition of the vertex set of $\Gamma$ and that $B^*$ is a refinement of $B$. For adjacent $B, C \in B$ and $\{\alpha, \beta\} = B \setminus \Gamma(C)$ as above, $\alpha$ and $\beta$ belong to different parts of $\Omega$, and so we may assume that $\alpha \in B_1$ and $\beta \in B_2$ without loss of generality. Since $t < v$, each part of $\Omega$ has size at least two and hence we can take a vertex $\xi \in B^2 \setminus \{\beta\}$. Since $t > 2$, $\Omega$ has at least three parts and so we can take a vertex $\eta \in B^3$. Since $B \setminus \Gamma(C) = \{\alpha, \beta\}$ and $\xi, \eta \neq \alpha, \beta$, each of $\xi$ and $\eta$ has at least one neighbour in $C$. Let $\xi$ be adjacent to $\gamma \in C$ and $\eta$ adjacent to $\delta \in C$. Since $\Gamma$ is $G$-symmetric, there exists an element $g \in G$ which maps $(\eta, \delta)$ to $(\xi, \gamma)$. Thus $g \in G_{BC}$. Since $B^*$ is $G$-invariant and $g$ maps $\eta \in B^3$ to $\xi \in B^2$, $g$ should map $B^3$ to $B^2$. Since every vertex in $B^3$ has a neighbour in $C$, it follows that every vertex in $B^2$ has a neighbour in $C$. However, this is a contradiction since $\beta \in B^2$ has no neighbour in $C$. Therefore we have proved Claim 4.

Obviously, if $\Omega \cong K_{v, \nu}$, then $d = v - 1, b = mdv/2 = m(v - 1)v/2$, and moreover $G_B$ is $2$-homogeneous on $B$ since $\Omega$ is $G_B$-edge-transitive by [2, Theorem 2.1].  

In the case $\Omega \cong K_{v/2, v/2}$, we have $d = v/2, b = mdv/2 = mv^2/4$, and the $G$-invariant partition $B^*$ above becomes $B^* = \{D_1, D_2 : D \in B\}$. Obviously, $B^*$ is a nontrivial partition of the vertex set of $\Gamma$ and is a refinement of $B$. In the case where each of $\Gamma(B^1)$ and $\Gamma(B^2)$ has nonempty intersection with exactly one of $C^1$ and $C^2$, it is easy to see that $v^* = k^* + 1, b = b^*, r = r^*$ and $s = s^*$, and so case (b)(i) occurs. In the remaining case, each of $\Gamma(B^1)$ and $\Gamma(B^2)$ has nonempty intersection with both $C^1$ and $C^2$, and hence $b^* = 2b$. If further every vertex in $B^1 \setminus \{\alpha\}$ has neighbours in both $C^1$ and $C^2$, then $v^* = k^* + 1, r^* = 2r$ and $s^* = s/2$, and so case (b)(ii) occurs. If not every vertex in $B^1 \setminus \{\alpha\}$ has neighbours in both $C^1$ and $C^2$, then by symmetry the numbers of vertices in $B^1 \setminus \{\alpha\}$ having neighbours in $C^1$ and $C^2$ are equal. This implies that

$$k^* = (v^* - 1)/2, \quad r^* = b^*k^*/v^* = b(v - 2)/v = r \quad \text{and} \quad s^* = rs/r^* = s,$$

and hence case (b)(iii) occurs.  

Example 2.4 in [2] can serve as an example for case (a) in Theorem 2 when $v = 3$. Examples for case (b)(i) when $v = 4$ can be obtained from [4, Construction 3.2]: let $M$ be a regular map on a closed surface such that its underlying graph $\Sigma$ has valency four. (A regular map is a 2-cell embedding of a connected (multi)graph on a closed surface such that its automorphism group is regular on incident vertex–face triples.) For each edge $\{\sigma, \sigma'\}$ of $\Sigma$, let $f$ and $f'$ denote the faces of $M$ with $\{\sigma, \sigma'\}$ as a common edge. Denote by $f_\sigma$ and $f'_\sigma$ the other two faces of $M$ incident with $\sigma$ and opposite to $f$ and $f'$ respectively, and define $f_\sigma'$ and $f'_\sigma'$ similarly. Let $\Gamma_1(M), \Gamma_2(M), \Gamma_3(M)$ and $\Gamma_4(M)$ be the graphs [4] with vertices the incident vertex–face pairs of $M$ and
adjacency defined as follows (where $\sim$ means adjacency): for each edge $\{\sigma, \sigma'\}$ of $\Sigma$, $(\sigma, f) \sim (\sigma', f)$ and $(\sigma, f') \sim (\sigma', f')$ in $\Gamma_1(M)$; $(\sigma, f) \sim (\sigma', f')$ and $(\sigma, f') \sim (\sigma', f)$ in $\Gamma_2(M)$; $(\sigma, f_\sigma) \sim (\sigma', f_\sigma')$ and $(\sigma, f_\sigma') \sim (\sigma', f_\sigma)$ in $\Gamma_3(M)$; $(\sigma, f_\sigma) \sim (\sigma', f_\sigma')$ and $(\sigma, f_\sigma') \sim (\sigma', f_\sigma')$ in $\Gamma_4(M)$. These graphs are $G$-symmetric [4, Lemma 3.3] and admit $B := \{B(\sigma) : \sigma \in V(\Sigma)\}$ as a $G$-invariant partition, where $B(\sigma) = \{(\sigma, f) : \sigma \text{ incident with } f\}$, such that $k = v - 2 = 2$, $\Gamma_B \cong \Sigma$, $\Gamma^{B(\sigma)} = K_{2,2}$ and $\Gamma[B(\sigma), B(\tau)] = 2 \cdot K_2$ for adjacent $B(\sigma), B(\tau) \in B$. These four graphs fall into case (b)(i) in Theorem 2 and the $G$-invariant partition induced by the bipartition of $\Gamma^{B(\sigma)}$ is $B^* := \{B^1(\sigma), B^2(\sigma) : \sigma \in V(\Sigma)\}$, where $B^1(\sigma) = \{(\sigma, f), (\sigma, f_\sigma)\}$ and $B^2(\sigma) = \{(\sigma, f'), (\sigma, f_\sigma')\}$.

References


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