Note
Upper bounds for $f$-domination number of graphs

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Abstract

For an integer-valued function $f$ defined on the vertices of a graph $G$, the $f$-domination number $\gamma_f(G)$ of $G$ is the smallest cardinality of a subset $D \subseteq V(G)$ such that each $x \in V(G) - D$ is adjacent to at least $f(x)$ vertices in $D$. When $f(x) = k$ for all $x \in V(G)$, $\gamma_f(G)$ is the $k$-domination number $\gamma_k(G)$. In this note, we give a tight upper bound for $\gamma_f$ and an improvement of the upper bound for a special $f$-domination number $\mu_{f,k}$ of Stracke and Volkmann (1993). Some upper bounds for $\gamma_k$ are also obtained. © 1998 Elsevier Science B.V. All rights reserved

Let $G = (V(G), E(G))$ be a finite, undirected, simple graph. The domination number of $G$, denoted by $\gamma(G)$, is the smallest cardinality of a set $D \subseteq V(G)$ such that each $x \in V(G) - D$ is adjacent to at least one vertex in $D$. Extensive studies on $\gamma(G)$ and domination-related topics have been done in the last thirty years. In 1985, Fink and Jacobson [4,5] introduced the concept of $k$-domination. For a positive integer $k$, a set $D \subseteq V(G)$ is called a $k$-dominating set if each $x \in V(G) - D$ is adjacent to at least $k$ vertices of $D$. The $k$-domination number $\gamma_k(G)$ is then defined to be the smallest cardinality of a $k$-dominating set of $G$ (see [4]). The following upper bound for $\gamma_k$ was proved in [1].

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**Theorem 1** (Caro and Roditty [1]). Let $G$ be a graph of $p$ vertices and the minimum degree $\delta(G) \geq ((n+1)/n)k - 1$, where $n$ and $k$ are positive integers. Then

$$\gamma_k(G) \leq \frac{n}{n+1} p. \quad (1)$$

This theorem generalizes the result that $\gamma_k(G) \leq kp/(k+1)$ if $\delta(G) \geq k$ (see [2]). In [6], a more general domination concept was introduced. For an integer-valued function $f$ defined on $V(G)$, a set $D \subseteq V(G)$ is called an $f$-dominating set of $G$ if each $x \in V(G) - D$ is adjacent to at least $f(x)$ vertices in $D$. The $f$-domination number $\gamma_f(G)$ is defined to be the smallest cardinality of an $f$-dominating set of $G$ (see [8]). For integers $j,k$ with $0 \leq j \leq k$, Stracke and Volkmann [6] defined the function $f_{j,k}(x) = \min\{j, j - k + d(x)\}$, where $d(x)$ is the degree of vertex $x$ in $G$. Then they studied the $f_{j,k}$-domination number $\mu_{j,k}(G)$ and obtained the following result.

**Theorem 2** (Stracke and Volkmann [6]). If $G$ is a graph of $p$ vertices and $0 \leq j \leq k$, then

$$\mu_{j,k}(G) \leq \begin{cases} \frac{2j - k}{2j - k + 1} p, & \text{if } j \leq k \leq 2j - 2, \\ p/2, & \text{if } k \geq 2j - 1. \end{cases} (2)$$

In this note we first generalize Theorem 1 to the case of $f$-domination number. With this generalization we then give an upper bound for $\mu_{j,k}$ which improves (2) slightly. As consequences, we obtain some upper bounds for $\gamma_k$. First we have the following theorem of which a weaker form appeared in [8].

**Theorem 3.** Let $f$ be an integer-valued function defined on $V(G)$ and let $n$ be a positive integer. If $f(x) < n/(n+1)(d(x) + 1 + 1/n)$ for each $x \in V(G)$, then

$$\gamma_f(G) \leq \frac{n}{n+1} p. \quad (3)$$

**Proof.** The proof applies a similar idea used in [1]. Set

$$v = \max_{x \in V(G)} \left( ((n+1)f(x) - n(d(x) + 1)) \right).$$

Then

$$d(x) \geq \frac{n+1}{n} f(x) - 1 - \frac{v}{n}, \quad (4)$$

and the given inequality implies $v < 1$.

Let $V_1, V_2, \ldots, V_{n+1}$ be a partition of $V(G)$ such that $H = G - \bigcup_{i=1}^{n+1} E(G[V_i])$ contains as many edges as possible, where $G[V_i]$ is the subgraph of $G$ induced by $V_i$. Let $d_H(x)$ denote the degree of $x$ in $H$. Then $d_H(x) \geq [(n/(n+1))d(x)]$ for each $x \in V(G)$.
(see [3], an explicit proof can be found in [7, pp. 233]). In fact, suppose to the contrary that \((n + 1)\delta(x) < nd(x)\) for a vertex, say, \(x \in V_1\). Let \(l \geq 2\) be such that the number of vertices in \(V_l\) which are adjacent to \(x\) is as small as possible. Let \(W_1 = V_1 - \{x\}\), \(W_l = V_l \cup \{x\}\) and \(W_i = V_i\), \(i \neq 1, l\). Then \(G - \bigcup_{i=1}^{n+1} E(G[W_i])\) has more edges than \(H\), a contradiction. From (4) we have

\[
d_H(x) \geq \left[ \frac{n}{n + 1} \left( \frac{n + 1}{n} f(x) - 1 - \frac{v}{n} \right) \right]
\]

\[
= \left[ f(x) - \frac{n + v}{n + 1} \right]
\]

\[
= \begin{cases} 
  f(x), & \text{if } -n \leq v < 1, \\
  > f(x), & \text{otherwise}.
\end{cases}
\]

Without loss of generality we may assume \(|V_1| = \max_{1 \leq i \leq n+1} |V_i|\). From the inequality above we know that \(V(G) - V_1\) is an \(f\)-dominating set of \(G\). Thus,

\[
\gamma_f(G) \leq p - |V_1| \leq p - \frac{p}{n+1} = \frac{n}{n+1} p. \quad \square
\]

**Corollary 4.** Let \(A\) be a subset of \(V(G)\) with \(\delta(G[A]) \geq 1\). Let

\[
n_0 = \begin{cases} 
  \frac{\delta(G[A])}{\delta(G[A]) - k + 1}, & \text{if } (\delta(G[A]) - k + 1)|(k - 1) \\
  \left[ \frac{k - 1}{\delta(G[A]) - k + 1} \right], & \text{otherwise}
\end{cases}
\]

for each \(k\) with \(1 \leq k \leq \delta(G[A])\). Then \(\gamma_k(G) \leq p - |A|/(n_0 + 1)\).

**Proof.** Since \(n_0 > (k - 1)/(\delta(G[A]) - k + 1)\), we have

\[
k < \frac{n_0}{n_0 + 1} \left( \delta(G[A]) + 1 + \frac{1}{n_0} \right).
\]

Hence, \(\gamma_k(G[A]) \leq (n_0/(n_0 + 1))|A|\) by Theorem 3. Since a minimum \(k\)-dominating set of \(G[A]\) together with \(V(G) - A\) yields a \(k\)-dominating set of \(G\), we get

\[
\gamma_k(G) \leq p - |A| + \gamma_k(G[A]) \leq p - \frac{|A|}{n_0 + 1}. \quad \square
\]

Theorem 3 is a generalization of Theorem 1, and the upper bound in (3) is attainable. For example, let \(x_0\) be a fixed vertex of the complete graph \(K_p\). Let \(f(x_0) = p - 2\) and \(f(x) = p - 1\) for all \(x \in V(K_p) - \{x_0\}\). One can easily check that for \(n = p - 1\) the condition in Theorem 3 holds, and it follows from (3) that \(\gamma_f(K_p) \leq p - 1\). In fact, \(\gamma_f(K_p) = p - 1\). Using Theorem 3 we can prove the following:
Theorem 5. Let $j, k$ be integers such that $0 \leq j \leq k$. Then

$$
\mu_{j,k}(G) \leq \begin{cases} 
\frac{2j - k - 1}{2j - k} p, & \text{if } j + 1 \leq k \leq 2j - 3, \\
\frac{k}{k + 1} p, & \text{if } k = j, \\
\frac{5}{3} p, & \text{if } k = 2j - 2, \\
\frac{1}{3} p, & \text{if } k \geq 2j - 1.
\end{cases}
$$

(5)

Proof. If $k \geq 2j - 1$, it was proved in [6] (also implied in Corollary 4 of [8]) that $\mu_{j,k}(G) \leq \frac{1}{2} p$. For the case $j + 1 \leq k \leq 2j - 3$, we claim that

$$
f_{j,k}(x) \leq \frac{2j - k - 1}{2j - k} \left( d(x) + 1 + \frac{1}{2j - k - 1} \right).
$$

(6)

We divide this into two cases.

Case 1: $d(x) \geq k$. Then $f_{j,k}(x) = j$ and (6) becomes $j(2j - k) < (2j - k - 1)(d(x) + 1) + 1$. To prove this, it suffices to show $j(2j - k) < (2j - k - 1)(k + 1) + 1$, or, equivalently, to show

$$
\left( k - \frac{3j - 2}{2} \right)^2 < \frac{1}{4}(j - 2)^2,
$$

which is true since $j < k < 2j - 2$.

Case 2: $d(x) \leq k - 1$. Then $f_{j,k}(x) = j - k + d(x)$ and (6) is equivalent to $d(x) + 1 < (k - j + 1)(2j - k) + 1$. To prove this, it suffices to check $k < (k - j + 1)(2j - k) + 1$, which is equivalent to

$$
\left( k - \frac{3j - 2}{2} \right)^2 < \frac{1}{4}(j - 2)^2 + 1.
$$

But this is true as we have proved earlier. Thus, (6) is valid provided that $j + 1 \leq k \leq 2j - 3$. From Theorem 3 we get

$$
\mu_{j,k}(G) \leq \frac{2j - k - 1}{2j - k} p.
$$

By a similar discussion as above it can be easily shown that $f_{j,2j-2}(x) \leq \frac{5}{3}(d(x) + 1)$ and $f_{k,k}(x) \leq (k/(k + 1))(d(x) + 1)$ for each $x \in V(G)$. Again, we get $\mu_{j,2j-2}(G) \leq \frac{7}{3} p$ and $\mu_{k,k}(G) \leq (k/(k + 1))p$ from Theorem 3. This completes the proof. □

Note that although (5) is just slightly better than (2) when $j + 1 \leq k \leq 2j - 3$, the proof is simpler. Theorem 5 implies the following improvement of Theorem 2 of [6].
Corollary 6. Let \( k \) and \( l \) be integers with \( 1 \leq k \leq l \) and let \( A_l = \{ x \in V(G) : d(x) \geq l \} \). Then
\[
\gamma_k(G) \leq p - \max \left\{ \max_{k+1 \leq i \leq 2k-3} \frac{|A_l|}{2k-l'}, \frac{|A_k|}{k+1}, \frac{|A_{2k-2}|}{3}, \frac{|A_{2k-1}|}{2} \right\}. \tag{7}
\]

Proof. Similar to the proof of Theorem 2 of [6]. \( \square \)

If \( l \leq \delta(G) \), then \( |A_l| = p \). So (7) implies

Corollary 7. For any integer \( k \geq 1 \),
\[
\gamma_k(G) \leq \begin{cases} 
\frac{2k - \delta(G) - 1}{2k - \delta(G)} \cdot p, & \text{if } k + 1 \leq \delta(G) \leq 2k - 3, \\
\frac{\delta(G)}{\delta(G) + 1} \cdot p, & \text{if } \delta(G) = k, \\
\frac{2}{3} p, & \text{if } \delta(G) = 2k - 2 \geq 2, \\
\frac{1}{2} p, & \text{if } \delta(G) \geq 2k - 1.
\end{cases} \tag{8}
\]

This is an improvement of Corollary 2 in [6].

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References