Henry Segerman — *Research Statement*

segerman@unimelb.edu.au · +61 428001729 · http://www.ms.unimelb.edu.au/~segerman/

My research is in the areas of 3-dimensional topology and hyperbolic geometry, specifically in tackling concrete problems, often from a combinatorial point of view.

1. Background

One of the overarching themes in 3-dimensional geometry and topology, and much of my work, is to link the combinatorial description of a 3-manifold with its geometric properties. My work has centred on understanding the geometry of 3-manifolds via triangulations. That is, we describe a manifold M by cutting it into tetrahedra, and record the combinatorial information of how those tetrahedra are glued to each other to obtain M. Then we describe some aspect of the geometry of the manifold by associating extra information to each tetrahedron. Two important examples that I have worked on are:

(1) The *deformation variety*: Given an ideal triangulation[1] of a manifold M with torus boundary components, we describe the shape of each tetrahedron as an ideal tetrahedron in 3-dimensional hyperbolic space ($\mathbb{H}^3$), subject to consistency conditions so that the the tetrahedra fit together properly around edges of the triangulation. Each shape can be described by a single complex number assigned to an edge of the tetrahedron, and the consistency conditions can be expressed as polynomial equations in these 'complex dihedral angles'[2]. The resulting variety is called the deformation variety and was introduced by Thurston [20]. Each point of the variety gives a (generally incomplete) hyperbolic structure on the manifold.

(2) The space of *angle structures*: This is closely related to the deformation variety. Again we are given an ideal triangulation. An angle structure records a dihedral angle in $[0, \pi]$ at each edge of each tetrahedron, subject to the condition that the sum of the angles around each edge must be 2π. If a point of the deformation variety has all tetrahedron shapes positively oriented, then the arguments of the complex dihedral angles give an angle structure. The equations for the deformation variety are non-linear. In contrast, the angle structure conditions are linear, and the space of angle structures is a convex polytope.

2. Special triangulations

2.1. Triangulations admitting strict angle structures.

It is conjectured that every hyperbolic 3-manifold with torus boundary components has a *geometric* triangulation, meaning that the complete hyperbolic structure[3] for the manifold is realised by gluing together positively oriented ideal hyperbolic tetrahedra. Equivalently, the complete hyperbolic structure is given by a point of the deformation variety at which every tetrahedron shape has positive imaginary part. This was believed to be an easy consequence of results of Epstein and Penner [5], but eventually recognised as a difficult problem (see [13] for a discussion).

In our most recent paper [6], Craig Hodgson, Hyam Rubinstein and I investigate constructions of triangulations admitting a *strict* angle structure, meaning that all angles are in $(0, \pi)$. A strict angle structure is a necessary condition for a triangulation to be geometric. Our main result is:

Theorem. *Assume that M is a cusped hyperbolic 3-manifold homeomorphic to the interior of a compact 3-manifold $\overline{M}$ with torus or Klein bottle boundary components. If $H_1(\overline{M}; \mathbb{Z}_2) \to H_1(\overline{M}, \partial\overline{M}; \mathbb{Z}_2)$ is the zero map then M admits an ideal triangulation with a strict angle structure.*

This is a relatively mild hypothesis. For example, it is satisfied by any knot or link complement in S^3. Although the triangulations we build will not generally be geometric, triangulations derived from them, or made using similar constructions may be.

[1] In an ideal triangulation the tetrahedra are missing their vertices, and we approach the boundary of the manifold as we move out towards the vertices.

[2] The complex number can also be seen as a cross-ratio of the four ideal vertices on the boundary of $\mathbb{H}^3$, viewed as $\mathbb{C} \cup \{\infty\}$.

[3] By results of Mostow and Prasad, only a single point of the deformation variety gives a complete metric.

segerman@unimelb.edu.au · +61 428001729 · http://www.ms.unimelb.edu.au/~segerman/

2.2. **Veering triangulations.** In another recent project, Craig Hodgson, Hyam Rubinstein, Stephan Tillmann and I investigated a particularly special class of triangulations, "veering triangulations", introduced by Ian Agol in 2010. A veering triangulation is a triangulation with a taut angle structure (meaning that the angles are all either 0 or π, so we can think of the tetrahedra as being flat), with an additional condition that the edges of the triangulation can be coloured in one of two colours so that the induced colouring on each tetrahedron has a particular form. In [8] we show:

Theorem. *Veering triangulations admit strict angle structures.*

The existence of a strict angle structure on some triangulation implies that the manifold admits a complete hyperbolic structure, by results of Casson and Thurston (see [10]). Thus the combinatorics of veering triangulations are related to the geometry of their manifolds, and the hope is that they provide a "good" choice of triangulation to which other geometric properties are closely related.

Agol [1] shows that a large class[4] of manifolds have veering triangulations. Using *Regina* [2] and its python scripting interface I searched within the SnapPea census [22] for manifolds with veering triangulations, using randomised retriangulation moves. Within the ≤ 5 tetrahedron census alone (a total of 301 orientable manifolds), I find 53 manifolds with some veering triangulation, and at least 6 of those cannot come from Agol's construction since the corresponding manifold does not fibre. The actual number of veering triangulations that do not come from Agol's construction is likely to be significantly higher, since the triangulation might not be layered even if the manifold does fibre. We find that veering triangulations are not all canonical (dual to the Ford domain), nor are they all minimal (with the minimum number of tetrahedra).

This is a very new and exciting area. It seems plausible that we could show that veering triangulations are indeed geometric, and there are many questions around how and under what conditions we can generate veering triangulations, and how general they are.

3. Degenerating geometric structures via the deformation variety

Much of my previous work has centred on understanding the deformation variety, particularly the relation between its *ideal points* and certain surfaces within the manifold. As we deform the geometric structure on a manifold (for example by following a path in the deformation variety, *representation variety*, or *character variety*[5]) the length of some path γ through the manifold can diverge to infinity. There is a well-known correspondence between points of the variety at which this behaviour occurs (these are the *ideal points* of the variety) and incompressible surfaces[6] within the manifold: The manifold splits apart along surfaces through which γ passes as the distances between regions of the manifold to each side of the surface diverge. This correspondence was introduced by Culler and Shalen in a seminal series of papers, giving a method to construct incompressible surfaces from ideal points of the character variety. The question naturally arises as to the extent that all incompressible surfaces come from ideal points. Not much is known about this question in general, although it is believed that "most" incompressible surfaces in "large" manifolds should not be detectable (i.e. constructible from ideal points).

[4] Namely, pseudo-Anosov mapping tori for which the singularities of the invariant foliations are at punctures.

[5] The representation variety is the set of representations from the fundamental group of the manifold into the isometries of $\mathbb{H}^3$. The character variety is, roughly, the set of traces of elements of the representation variety. Unlike the deformation variety, these do not depend on a choice of ideal triangulation of the manifold.

[6] Incompressible surfaces within a 3-manifold are intrinsic to the topological structure of the 3-manifold, and are an important tool in the study of 3-manifolds.

segerman@unimelb.edu.au · +61 428001729 · http://www.ms.unimelb.edu.au/~segerman/

My thesis work, and my paper [15] answered this question in the context of punctured torus bundles[7], and also gave a concrete geometric understanding of the splittings of these manifolds. The main result is:

Theorem. *All incompressible and boundary incompressible surfaces in hyperbolic punctured torus bundles over the circle can be constructed from ideal points of the deformation variety.*

I used an alternate version of the Culler-Shalen construction [4] due to Yoshida [23]. Starting from an ideal point of the deformation variety, Yoshida constructs a surface from twisted squares (see Figure 1(a)) placed in tetrahedra that degenerate as we approach the ideal point. We would also like to be able to say that the ideal points of the deformation variety that we find correspond to ideal points of the representation variety, and that the twisted squares surfaces we generate using Yoshida's construction can also be generated from the Culler-Shalen construction. Tillmann [21] proves this in general for spun-normal surfaces[8] generated from ideal points of the deformation variety.

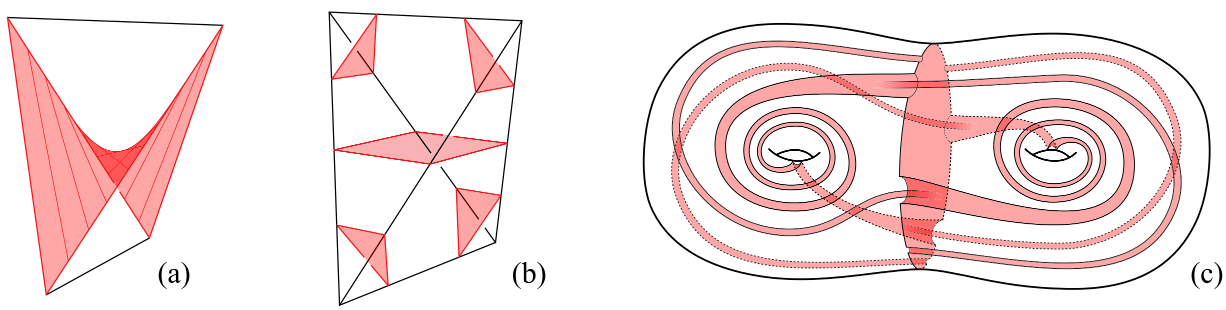

FIGURE 1. Twisted squares and spun-normal surfaces are two kinds of combinatorially defined surfaces embedded in a manifold with triangulation. The former are made up of twisted squares, shown in (a), the latter of quadrilateral and triangle parts, shown in (b). An example of an incompressible surface in a handlebody is shown in (c).

In [17] I investigated the relationship between twisted squares surfaces and spun-normal surfaces, and using Tillmann's analogous result for spun-normal surfaces, showed the following:

Theorem. *Let M be an oriented 3-manifold with boundary a union of tori with given ideal triangulation, and T a two-sided twisted squares surface obtained via Yoshida's construction from an ideal point of the deformation variety which corresponds to an ideal point of the character variety. Then any incompressible surface obtained from T by compressions is detected by the character variety.*

This paper then extends the result of [15] from ideal points of the deformation variety and Yoshida's construction to ideal points of the character variety and the Culler-Shalen construction.

3.1. **Problems with the deformation variety.** The deformation variety is a more concrete and visual object than the representation and character varieties, but has some problematic features. In particular, the deformation variety depends crucially on the chosen triangulation of the manifold: there may be entire components of the deformation variety with one triangulation for which no corresponding component exists in the deformation variety with a second triangulation.

In particular, the proof of [15] goes through by demonstrating that each incompressible surface has an ideal point of the deformation variety corresponding to it. It was a stroke of luck that the triangulation I used for each punctured torus bundle was such that the deformation variety with respect to that triangulation had

[7]Punctured torus bundles are a well studied class of three-manifolds with well understood triangulations.
[8]Spun-normal surfaces are made up of disks of the forms shown in Figure 1(b).

segerman@unimelb.edu.au · +61 428001729 · http://www.ms.unimelb.edu.au/~segerman/

all of the required ideal points. If I had been unable to find corresponding ideal points we would not know if there were no such ideal point to be found (as can happen in the case of 2-bridge knots [12]), or if we had just made a bad choice of triangulation. This issue, and the difficulty in choosing a "good" triangulation ahead of time led to the following project.

3.2. **The extended deformation variety.** In [16], I introduced a generalisation of the deformation variety, which again consists of assignments of complex variables to certain edges of the tetrahedra, subject to polynomial equations, but together with some extra combinatorial data concerning degenerate tetrahedra. This "extended deformation variety" deals with many situations that the deformation variety cannot. To a large extent it solves the problem of dependence on the triangulation, and should be very useful in making the deformation variety more like the representation variety, while preserving the visual and combinatorial aspects of working with triangulations. The main result is:

Theorem. *Let M be a connected 3-manifold with non-empty boundary consisting of a disjoint union of tori. Then there exists an ideal triangulation of M such that every irreducible representation whose image is not a generalised dihedral group can be recovered from the associated extended deformation variety.*

Tillmann [21] notes that if M admits a complete hyperbolic structure of finite volume and a (standard) deformation variety for M with some triangulation is non-empty, then it has a component which corresponds to the component of the representation variety containing the complete structure. However, individual representations may be missing from this correspondence, and it says nothing about other components. The extended deformation variety fills these gaps, acting in some ways like a compactification. I also show that the extended deformation variety detects the $PSL_2(\mathbb{C})$ A-polynomial[9].

3.3. **Future directions.** The extended deformation variety solves the problem of not knowing if the triangulation is bad or not, and so an obvious next step to take is to figure out how to compactify the extended deformation variety, in a way similar to how Yoshida [23] and Tillmann [21] compactify the (standard) deformation variety. This should again lead to a construction of incompressible surfaces. Closed incompressible surfaces are particularly difficult to identify using existing techniques (neither Tillmann nor Yoshida tackles them), but this work should provide a different approach which does not suffer from these issues.

An alternative approach would be to use a triangulation that is good for finding ideal points in the first place. Veering triangulations may be good in this sense: the simplest examples of veering triangulations are the canonical triangulations of punctured torus bundles, which are precisely the triangulations that I used in [15]. Certain features of these triangulations that were essential to understanding the ideal points of their deformation varieties have analogues in veering triangulations, and so the techniques may generalise.

4. Dehn surgery space

Another way of organising geometric structures in the case of a hyperbolic 3-manifold M with a single torus boundary is the Dehn surgery space of M. See Figure 2[10]. The Dehn surgery space can be seen as a subset of $\mathbb{C} \cup \{\infty\}$, with points corresponding to geometric structures with particular behaviours on the boundary torus. The point at ∞ corresponds to the complete structure, and the geometry corresponding to an integral point (x, y) is such that moving x times around the meridian and y times around the longitude of the boundary torus is trivial. There is a way to extend to non-integral points. By a result of Thurston [20], Dehn surgery space contains a neighbourhood of ∞.

A current project with Saul Schleimer is to understand the relationship between Dehn surgery space and the deformation variety. In particular, there is a map from the universal cover of the deformation variety

[9] Strictly speaking, it detects all factors apart from those coming from components consisting of reducible representations or representations with image a generalised dihedral group.

[10] Also see http://www.ms.unimelb.edu.au/~segerman/dehn_surgery_images.html for many more examples.

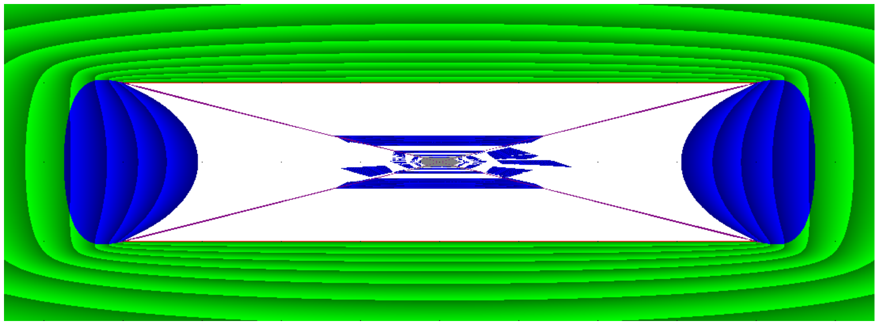

FIGURE 2. Computed picture showing the boundary of Dehn surgery space for the figure 8 knot complement. Tracy Hall, Saul Schleimer and I generated this image using SnapPeaPython [3] by Marc Culler and Nathan Dunfield, who built on SnapPea [22] by Jeff Weeks. The green and blue areas are hyperbolic structures with the shading corresponding to volume, blue with some negatively oriented tetrahedra and green with all positively oriented tetrahedra. Other colours are outside of Dehn surgery space. The straight lines on the boundary consist of $\widetilde{SL_2\mathbb{R}}$ structures (and Sol structures at $(0, \pm 1)$).

to Dehn surgery space, rather from the deformation variety itself. Under a smoothness assumption on this map, we can show that the "flat" geometric structures, where all tetrahedra are zero volume, form straight lines in Dehn surgery space. In addition, each line segment corresponds to a taut angle structure on the triangulation. We give a necessary condition on a taut angle structure for it to correspond to a line in Dehn surgery space, and show that veering structures do not satisfy this condition.

5. Other Projects

5.1. **Essential edges.** A project with Stephan Tillmann [19], filled an overlooked gap in the literature: A common move (first used in [23]) is to use a map from the deformation variety to the representation variety, or to the character variety. A key step in the definition of these maps is to construct a *developing map* for a given point of the deformation variety, from the universal cover of the manifold into $\mathbb{H}^3$. This step is only possible if each edge in the triangulation is *essential*, meaning that it cannot be homotoped into the boundary of the manifold. Previously authors have implicitly assumed that all edges are essential. We show:

Theorem. *Let M be a topologically finite, orientable 3–manifold with ideal triangulation. If the deformation variety is non-empty, then all edges in the triangulation are essential.*

We also extend this result to a variant of the deformation variety, which enables the construction of hyperbolic cone-manifold structures on M with singular locus contained in the 1-skeleton of the triangulation.

Craig Hodgson, Hyam Rubinstein, Stephan Tillmann and I are working on a related project [7], giving constructions of essential triangulations from various topological properties. We consider both closed manifolds and those with torus boundary components, and also the condition of being *strongly essential*, meaning that in addition to being essential, no two edges are homotopic fixing their endpoints.

5.2. **Incompressible surfaces in handlebodies.** A project [11] with João Miguel Nogueira studied incompressible (but not boundary incompressible) surfaces in the genus two handlebody. We show that for every compact surface with boundary, orientable or not, there is an incompressible embedding of the surface into the genus two handlebody. See Figure 1(c) for an example. In the orientable case the embedding can be either separating or non-separating. We extend a result of Qiu [14], who shows the existence of orientable incompressible surfaces with any genus but only two boundary components. We also consider the case in

segerman@unimelb.edu.au · +61 428001729 · http://www.ms.unimelb.edu.au/~segerman/

which the genus two handlebody is replaced by an orientable 3-manifold with a compressible boundary component of genus at least two.

5.3. **Neuroscience.** I am working with a group of neuroscientists led by Sukant Khurana. Using an overhead camera, we track the positions of larvae as they explore a Petri dish with various stimuli. We use features of the path that the larvae follow to distinguish between different internal states of the larvae, and to build a model of the motion of the larvae to test theories of how they sense and navigate an environment. Our recent work [9] (submitted for review) is a theoretical paper analysing various numerical measures of "amount of learning" that biologists have employed.

5.4. **Mathematical art.** I have an ongoing interest in applications of mathematics to the arts. My most recent paper [18] generalises the constructions of space filling curves to sequences of fractal graphs, generated by iterated substitution. Steps in the constructions of space filling curves are a common subject of mathematical sculpture, and this generalisation was motivated by the lack of structural robustness in a physical representation consisting of a long polygonal curve. The idea is to improve robustness, in part by increasing the vertex degree, although this alone turns out to not be sufficient. The paper also introduced the Cheeger constant as a graph-theoretic measure of structural robustness of a physical model of a graph.

References

[1] Ian Agol, *Ideal triangulations of pseudo-Anosov mapping tori*, arXiv:1008.1606v2.

[2] Benjamin A. Burton, *Regina: Normal surface and 3-manifold topology software*, http://regina.sourceforge.net/, 1999–2009.

[3] Marc Culler and Nathan Dunfield, *SnapPeaPython*, Python interface to the SnapPea kernel.

[4] Marc Culler and Peter Shalen, *Varieties of group representations and splitting of 3-manifolds*, Annals of Mathematics **117** (1983), 109–146.

[5] D. B. A. Epstein and R. C. Penner, *Euclidean decomposition of non-compact hyperbolic manifolds*, J. Differential Geom. **27** (1988), 67–80.

[6] Craig D. Hodgson, J. Hyam Rubinstein and Henry Segerman, *Triangulations of hyperbolic 3-manifolds admitting strict angle structures*, arXiv:math/1111.3168.

[7] Craig D. Hodgson, J. Hyam Rubinstein, Henry Segerman and Stephan Tillmann, *Triangulations of 3-manifolds with essential edge loops* (in progress).

[8] ———, *Veering triangulations admit strict angle structures*, *Geometry & Topology* **15** (2011), 2073–2089.

[9] Sukant Khurana, Wen-ke Li, Henry Segerman and Nigel S. Atkinson, *Evaluation of fractional learning indices for associative conditioning*, preprint.

[10] Marc Lackenby, *Word hyperbolic Dehn surgery*, Inventiones Mathematicae **140** (2000), 243–282.

[11] João Miguel Nogueira and Henry Segerman, *Incompressible surfaces in handlebodies and boundary compressible 3-manifolds*, *Topology and its Applications* **158** (2011), no. 4, 551-571.

[12] Tomotada Ohtsuki, *Ideal points and incompressible surfaces in two-bridge knot complements*, J. Math. Soc. Japan **46** (1994), no. 1, 51–87.

[13] Carlo Petronio and Joan Porti, *Negatively oriented ideal triangulations and a proof of Thurston's hyperbolic Dehn filling theorem*, Expo. Math. **18** (2000), no. 1, 1–35.

[14] Ruifeng Qiu, *Incompressible surfaces in handlebodies and closed* 3*-manifolds of Heegaard genus* 2, Proc. Amer. Math. Soc. **128** (2000), no. 10, 3091–3097.

[15] Henry Segerman, *Detection of incompressible surfaces in hyperbolic punctured torus bundles*, Geometriae Dedicata, **150** (2011), no. 1, 181–232.

[16] ———, *A generalisation of the deformation variety*, arXiv:math/0904.1893.

[17] ———, *On spun-normal and twisted squares surfaces*, Proc. Amer. Math. Soc. **137** (2009), 4259–4273.

[18] ———, *Fractal graphs by iterated substitution*, *Journal of Mathematics and the Arts* **5** (2011), no. 2, 51–70.

[19] Henry Segerman and Stephan Tillmann, *Pseudo-developing maps for ideal triangulations I: Essential edges and generalised hyperbolic gluing equations*, arXiv:1107.1030, Proceedings of the Jacofest, AMS Contemporary Mathematics, in press.

[20] William Thurston, *Geometry and topology of 3-manifolds*.

[21] Stephan Tillmann, *Degenerations of ideal hyperbolic triangulations*, Mathematische Zeitschrift, in press; arXiv:math/0508295v3.

[22] Jeffrey Weeks, *SnapPea*, a computer program for creating and studying hyperbolic 3-manifolds, available from http://www.geometrygames.org/SnapPea/.

[23] Tomoyoshi Yoshida, *On ideal points of deformation curves of hyperbolic 3-manifolds with one cusp*, Topology **30** (1991), no. 2, 155–170.