Disjoint Hamiltonian Cycles in Fan 2k-Type Graphs

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ABSTRACT

It is conjectured that a 2(k + 1)-connected graph of order p contains k + 1 pairwise disjoint Hamiltonian cycles if no two of its vertices that have degree less than $\frac{1}{2}p + 2k$ are distance two apart. This is proved in detail for k = 1. Similar arguments establish the conjecture for k = 2. © 1993 John Wiley & Sons, Inc.

1. INTRODUCTION

We shall, for the most part, use the terminology of [1]. Graphs will be finite, simple, and undirected. Throughout the paper $G = (V, E)$ is a graph with vertex set $V$ and edge set $E$. For $u, v \in V$, let $d(u)$ and $d(u, v)$ be the degree of $u$ and the distance between $u$ and $v$ in $G$, respectively. A c-matching of $G$ is a matching containing c edges. For two disjoint subsets $S, T$ of $V$, let $\delta(S, T) = \{uv \in E | u \in S, v \in T\}$. Suppose $P = uv \cdots w$ and $Q = xy \cdots z$ are two vertex-disjoint paths of $G$; we denote by $PQ$ the path $uv \cdots wxy \cdots z$ if $wx \in E$. Two Hamiltonian cycles (H-cycles, for short) are called disjoint when they share no common edge, and similar terminology will be applied to disjoint paths. A Hamiltonian path (H-path) starting and ending at $u$ and $v$ is called an H $(u, v)$-path.

The following theorem due to Geng-hua Fan ([2]) is well known:

**Theorem.** If a 2-connected graph $G$ of order $p$ satisfies the condition $d(u, v) = 2 \Rightarrow \max\{d(u), d(v)\} \geq \frac{1}{2}p$, then $G$ contains an H-cycle.

The proof of Fan's result was simplified by Tian Feng ([3]). In this paper we discuss graphs such that $d(u, v) = 2$ implies $\max\{d(u), d(v)\} \geq \frac{1}{2}p + 2k$ for some fixed $k$.

Definition. For a nonnegative integer \( k \), a graph \( G \) of order \( p \) is called a Fan \( 2k \)-type graph if \( d(u, v) = 2 \) implies \( \max\{d(u), d(v)\} \geq \frac{1}{2}p + 2k \).

Our main result can be stated as follows:

Theorem 1. Every 4-connected Fan 2-type graph contains two disjoint \( H \)-cycles.

2. PROOF OF MAIN RESULT

Lemma 1. If \( G \) is a Fan 2-type graph of order \( p \) and \( u, v \) are nonadjacent vertices of \( G \) that satisfy \( \min\{d(u), d(v)\} \geq \frac{1}{2}p + 2 \), then

1. \( G + uv \) is also a Fan 2-type graph, and
2. \( G \) contains two disjoint \( H \)-cycles if and only if \( G + uv \) contains two disjoint \( H \)-cycles.

Proof. In fact, when \( x, y \) have distance two in \( G + uv \), then either they have the same distance in \( G \) or \( \{x, y\} \cap \{u, v\} \neq \emptyset \), so (1) is valid. For (2), let \( C_1, C_2 \) be two disjoint \( H \)-cycles of \( G + uv \). We will prove that \( G \) contains two disjoint \( H \)-cycles as well. If \( uv \notin E(C_1) \cup E(C_2) \), then \( C_1 \) and \( C_2 \) are the needed cycles of \( G \). If \( uv \) belongs to \( E(C_1) \) or \( E(C_2) \), say \( uv \in E(C_2) \), then \( G' = G - E(C_1) \) has an \( H \)-path \( C_2 - uv = x_1x_2 \cdots x_p \), where \( x_1 = u, x_p = v \). Let

\[
M = \{x_i | x_ix_i \in E(G'), 2 \leq i \leq p - 1\}, \\
N = \{x_i | x_{i-1}x_p \in E(G'), 3 \leq i \leq p\}.
\]

Then

\[|M| + |N| = d_{G'}(x_1) + d_{G'}(x_p) \geq p\]

and

\[p - 1 \geq |M \cup N|,
\]

where \( d_{G'}(x_1), d_{G'}(x_p) \) denote the degree of \( x_1, x_p \) in \( G' \), respectively. We have

\[|M \cap N| = |M| + |N| - |M \cup N| \geq 1.
\]

Hence, there exists \( x_i \in M \cap N \). This gives an \( H \)-cycle \( C_1' = x_i x_1 x_2 \cdots x_{i-1}x_p x_{p-1} \cdots x_{i+1}x_i \) of \( G' \) and the proof is complete.
Since the complete graph $K_p$ can be decomposed into $\frac{1}{2}p$ H-paths when $p$ is even or decomposed into $\frac{1}{2}(p - 1)$ H-cycles while $p$ is odd (see [4]), we have

**Lemma 2.** For any four distinct vertices $u, v, x, y$ of the complete graph $K_p$ ($p \geq 4$), there exist an H($u, v$)-path and an H-($x, y$) path that are disjoint.

**Proof of Theorem 1.** Let $G = (V, E)$ be a 4-connected Fan 2-type graph of order $p$. Of necessity $p \geq 5$, and if $p \leq 7$ then $\left\lfloor \frac{1}{2}p \right\rfloor + 2 \geq p - 1$ and $G$ must be complete. Since $G$ has two disjoint H-cycles in this case, we now assume $p \geq 8$. Let

$$S = \left\{ u \in V \mid d(u) \leq \frac{1}{2}p + 2 \right\}. $$

By hypothesis, $S \neq \emptyset$. Because of Lemma 1, we may assume that $G[S]$ is complete. In case $S = V$, we conclude that $G$ contains two disjoint H-cycles by applying Dirac's theorem twice. In the remainder of the proof, suppose $V \setminus S \neq \emptyset$ and let $G_i = (V_i, E_i) (1 \leq i \leq \omega)$ denote the components of $G[V \setminus S]$. Let

$$S_i = \{ u \in S \mid uv \in E \text{ for some } v \in V_i \}, \quad 1 \leq i \leq \omega.$$

By hypothesis, no two vertices of $V \setminus S$ are distance two apart. It follows that

**Claim 1.** (1) $G_i$ is complete, $1 \leq i \leq \omega$, and (2) $S_i \cap S_j = \emptyset, i \neq j$.

Let

$$T_i = \{ v \in V_i \mid uv \in E \text{ for some } u \in S_i \}, \quad 1 \leq i \leq \omega.$$

Then $T_i = V_i$ if $|T_i| \leq 3$ since a vertex in $V \setminus T_i$ can be disconnected from the rest of the graph by the deletion of $T_i$.

**Claim 2.** (1) $|S_i| \geq 4, 1 \leq i \leq \omega$, and hence (1) $|\partial(S_i, T_i)| \geq 4, 1 \leq i \leq \omega$.

Suppose to the contrary that $|S_i| \leq 3$ for some $i$. In view of the fact that $G$ is 4-connected, an immediate contradiction is reached unless $\omega = 1$ and $S = S_1$. In this case, since any vertex $v$ in $T_1$ must have degree at least $p - |S|$, we have

$$p - |S| \leq d(v) \leq \frac{1}{2}p + 1.$$
so
\[ p \leq 2(|S| + 1) \leq 8. \]

Since by assumption \( p \geq 8 \) we get \( p = 8 \), \( |S| = 3 \), \( |V \setminus S| = 5 \), and each vertex in \( T_1 \) has degree five. Because of Claim 1(1), there exists a vertex of \( S \) that is adjacent to at most one vertex in \( V \setminus S \), which conflicts with the 4-connectivity of \( G \). This ensures the validity of Claim 2.

Denote by \( B_i \) the bipartite graph \((S_i \cup T_i, \delta(S_i, T_i))\), \( 1 \leq i \leq \omega \). Using the previous claims, and recalling that for any bipartite graph the cardinality of its maximum matching is equal to the cardinality of its minimum vertex-covering, we get

**Claim 3.** \( B_i \) contains a matching of \( \min\{|V_i|, 4\} \) edges, \( 1 \leq i \leq \omega \).

Since \( \delta(G) \geq 4 \), we have

**Claim 4.** If \( |V_i| \leq 3 \) for some \( i \), then each \( v \in V_i \) adjacent to at least \( (5 - |V_i|) \) vertices of \( S_i \).

**Claim 5.** For each \( i \) between 1 and \( \omega \), the graph \( G[V_i \cup S_i] \) contains two disjoint H-paths \( P \) and \( P' \) such that

1. the end-vertices of \( P \) and \( P' \) are distinct and lie in \( S_i \), and
2. \( |E(P \cup P') \cap E(G[S_i])| \geq 3 \).

This claim can be proved case by case according to \( |V_i| = 1, 2, 3 \) or \( \geq 4 \). If \( |V_i| \geq 4 \), let \( \{xu, yv, x'u', y'v'\} \) be a 4-matching of \( B_i \) ensured by Claim 3, where \( x, y, x', y' \in T_i, u, v, u', v' \in S_i \). By Lemma 2, there is an H-(x, y) path \( Q \) and an H-\((x', y')\)-path \( Q' \) in \( G[V_i] \) such that \( E(Q) \cap E(Q') = \emptyset \).

By the same reason, there is an H-\((u, v)\)-path \( R \) and an H-\((u', v')\)-path \( R' \) in \( G[S_i] \) that are disjoint. Thus \( C = QRx \) and \( C' = Q'R'x' \) are two disjoint H-cycles of \( G[V_i \cup S_i] \). Choose two edges \( e \in E(C) \cap E(G[S_i]) \) and \( e' \in E(C') \cap E(G[S_i]) \) such that they have no common vertex. Then \( P = C - e \) and \( P' = C' - e' \) are the required H-paths of \( G[V_i \cup S_i] \) and (2) is valid. For \( |V_i| = 3 \), let \( V_i = \{u, v, w\} \). If \( |S_i| \geq 5 \), we can get the required H-paths by using the same method as above. If \( |S_i| = 4 \), then by Claim 3 we may suppose \( \{xu, yv, zw\} \) is a 3-matching of \( B_i \). For the situation that all \( u, v, w \) are adjacent to the fourth vertex \( z' \) of \( S_i \), \( yu'z'wuxz \) and \( xz\z'wuz'z \) are two disjoint H-paths of \( G[V_i \cup S_i] \), and (2) is apparently true. The readers may similarly verify the claim when other situations appear. So Claim 5 is true for \( |V_i| = 3 \). The remaining cases of the proof can be dealt with as above using Claims 1–4 and Lemma 2.

Now we construct two disjoint H-cycles of \( G \). For each \( i \), let \( P_i \) and \( P'_i \) be two disjoint H-paths of \( G[V_i \cup S_i] \) that satisfy (1) and (2) in Claim 5. Let \( u_i \) and \( u'_i \) be the starting vertices of \( P_i \) and \( P'_i \),
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respectively. Put

\[ S_0 = S \setminus \left( \bigcup_{i=1}^{\ell} S_i \right) . \]

If \(|S_0| = 0\), then \(P_1P_2\ldots P_\omega u_1\) and \(P'_1P'_2\ldots P'_\omega u'_1\) are the required \(H\)-cycles. If \(|S_0| = 1\), say \(S_0 = \{\xi\}\), then \(P_1P_2\ldots P_\omega \xi u_1\) and \(P'_1P'_2\ldots P'_\omega \xi u'_1\) are disjoint \(H\)-cycles of \(G\). For \(|S_0| = 2\), let \(S_0 = \{\xi, \xi'\} \). By Claim 5(2) without loss of generality, we can suppose the two edges \(\xi \eta, \xi' \eta' \in E(P_1) \cap E(G[S_1])\). Then \(P_1P_2\ldots P_\omega u_1, \) and \(P'_1P'_2\ldots P'_\omega \xi \xi' u'_1\) are the required \(H\)-cycles, where \(P_1\) is the path obtained by replacing \(\xi \eta\) by \(\xi \xi' \eta\) and \(\xi' \eta'\) by \(\xi' \xi' \eta'\) in \(P_1\). The case \(|S_0| = 3\) may be dealt with just as above. For the case \(|S_0| \geq 4\), Lemma 2 ensures the existence of two disjoint \(H\)-paths \(P_0\) and \(P'_0\) of \(G[S_0]\). Thus \(P_1P_2\ldots P_\omega P_0 u_1\) and \(P'_1P'_2\ldots P'_\omega P'_0 u'_1\) are two disjoint \(H\)-cycles of \(G\). This completes the proof of Theorem 1.

3. A FURTHER RESULT AND A CONJECTURE

Using the same techniques as in the last section, we get

**Theorem 2.** Every 6-connected Fan 4-type graph contains three pairwise disjoint \(H\)-cycles.

We conjecture the following more general result:

**Conjecture.** For \(k\) is a nonnegative integer, every \(2(k + 1)\)-connected Fan \(2k\)-type graph has \(k + 1\) pairwise disjoint \(H\)-cycles.

We conclude this article by pointing out that the \((2k + 1)\)-connectivity in the conjecture above, as well as in Theorems 1 and 2, is needed. For instance, consider the graph consisting of two vertex-disjoint complete graphs \(K_{2k+1}\) and \(K_p\) \((p \geq 6k + 3)\) and a \((2k + 1)\)-matching between the vertex sets of \(K_{2k+1}\) and \(K_p\). Evidently this graph is a \((2k + 1)\)-connected Fan \(2k\)-type graph, but it does not have \(k + 1\) pairwise disjoint \(H\)-cycles.

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