1st order Linear Diff. Eq.

The simplest case - constant coefficient

\[ y_n + ax_n = 0 \]  
(homogeneous)

\[ y = y_0 \]
\[ y = y_{0x} \]
\[ y = y_{0x}^2 \]

\[ y_n = a^x y_0 \]

By inspection,

- \( a > 1 \) \( \quad \Rightarrow \quad y_n \to \infty \)
- \( a = 1 \) \( \quad \Rightarrow \quad y_n = y_0 \)
- \( 0 < a < 1 \) \( \quad \Rightarrow \quad y_n \to 0 \)
- \( a = 0 \) \( \quad \Rightarrow \quad y_n = 0 \)
- \( -1 < a < 0 \) \( \quad \Rightarrow \quad y_n \to 0 \), oscillates
- \( a < -1 \) \( \quad \Rightarrow \quad y_n = \pm \infty \)
- \( a = \infty \) \( \quad \Rightarrow \quad y_n \to \infty \)

\[ y_n = a^x y_0 \]  
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Set of \( y_n = a^x y_0 \) \( \to 0 \) if \( \quad |a| < 1 \)

\[ a_1 \neq 1 \]

\[ y_n = a y_n + \beta \]

Recall: Step 1: \( y = y_0 \)
From above \( \quad y_0 = a^x \)
Step 2: Find a set \( y^* \) to the inhom. eq.
- any solution will do!

By far the easiest & fitted is a constant solution

\[ y^* = y_0 \quad \forall n \]  
(if it exists)

Let's try:

\[ y^* = y_0 = a^x y_0 + \beta \]
\[ (1 - a) y^* = \beta \]
\[ y^* = \frac{\beta}{1 - a} \]

\( \quad \Rightarrow \quad \) a constant solution \( \quad \text{if} \quad a \neq 1 \)
Step 3:
\[ a \neq 1 \]
\[ y_n = c a^n + \frac{\beta}{1 - a} \]

\( a = 1 \)

\[ y_n = a y_n + \beta \]
\[ y_0 = y_0 \quad \Rightarrow \quad y_n = y_0 \quad \forall n \]
\[ y_0 = y_0 + \beta \]

\( \Rightarrow \quad y_n = y_0 + \frac{\beta}{1 - a} \quad \text{iff} \quad |a| < 1 \)

Something harder...

\[ y_n = \frac{m}{m+2} y_{n+2} \quad \frac{m+1}{m+2} y_{n+1} \]
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\( \Rightarrow \quad y_n = \frac{m}{m+2} y_{n+2} \quad \frac{m+1}{m+2} y_{n+1} \]

We have \( \quad y^* = y_0 \)
Since \( \quad y_n \) is a function of \( n \), don't expect a constant solution
Try to find \( y^* \) - i.e. \( y_n = 0 \) since \( y^* \) will do!

\( \Rightarrow \quad \)
\( \quad \Rightarrow \quad \)

\( \Rightarrow \quad \)

\( \Rightarrow \quad \)

Prove by induction
Financial Math.

We have an account that contains \( A_0 \) dollars after \( n \) compounding periods. The account is collecting \( r \) percent per period.

In addition, a constant amount \( b \) dollars per period is added (or withdrawn if \( b < 0 \)).

Let \( A_n \) be the initial amount in the account:

\[
\begin{align*}
A_n &= A_0 \left(1 + \frac{r}{100}\right) + b \\
&= \left(1 + \frac{r}{100}\right)^n + \frac{b}{r} \\
&= \left(1 + \frac{r}{100}\right)^n - \frac{b}{r}
\end{align*}
\]

As \( \beta = 0 \), \( A_0 = A_n \):

\[
A_0 = c_0 - \frac{b}{r} \quad \Rightarrow \quad c_0 = A_0 + \frac{b}{r}
\]

\[
A_n = (A_0 + \frac{b}{r}) \left(1 + \frac{r}{100}\right)^n - \frac{b}{r}
\]

or

\[
A_n = A_0 \left(1 + \frac{r}{100}\right)^n + b \frac{(1 + \frac{r}{100})^n - 1}{\frac{r}{100}}
\]

An alternative form would be:

\[
FV = PV \left(1 + \frac{rate}{n}\right)^{n \times periods} + PMT \left(1 + \frac{rate}{n}\right)^{n \times periods} \times \frac{1 - \left(1 + \frac{rate}{n}\right)^{-n \times periods}}{\frac{rate}{n}}
\]

Warning: Need all the conventional.

Here \( A_0 > 0 \), \( b > 0 \).

In Excel, ensure you pay out are negative:

So, \( PMT = -b \)

\( PV = -A_0 \)

In Excel:

\[
FV + PV \left(1 + \frac{rate}{n}\right)^{n \times periods} + PMT \left(1 + \frac{rate}{n}\right)^{n \times periods} \times \frac{1 - \left(1 + \frac{rate}{n}\right)^{-n \times periods}}{\frac{rate}{n}}
\]

Just using a different convention of \( PV \) & \( PMT \).

If you’re given \( t \) of the total, you can calculate the remaining quantity.

Ats, graphic calculator/spreadheets implement these as Financial Functions.

Many simple financial problems are just applications of the formula to various financial products - annuities, annuities, loans etc.

Here are 5 variables appearing:

- \( A_n \) - the amount after \( n \) periods
- \( A_0 \) - the initial amount in the account
- \( i \) - the interest rate per period
- \( m \) - number of periods
- \( b \) - the amount of payment made into the account

\( r \) is 10% p.a. but compounded quarterly, \( n = 4 \times 12 = 48 \)

\( m = 3 \times 12 = 36 \)

\( b = 12 \) as \( rate = 10\% \) (no periodic payment)

So, Excel does the function:

\[
FV, \ PV, \ PMT, \ Rate, \ PV
\]

In each case, you need to provide the other 4 variables.

For \( rate \), also need a guess.

\[
-FV = PV \left(1 + \frac{Rate}{n}\right)^{n \times periods} = Compound \ Interest
\]

If you’re a millionaire, you need \( i \) deposit an amount \( I \) at rate \( r \) compounded quarterly per year on 25th birthday.

How much would you have in deposit if

\[
r = 10\% \quad (r = \% ? \quad 10\% ?)
\]
1. Identify which rates you know.
2. Identify which to find.

We are given $i = \frac{0.05}{10} = 0.005$
$v = 50 + 4 = 54$

We want to find the present value.

2. Rearrange formula:

$$PV = \frac{FV}{(1+i)^n} = \frac{10^6}{(1.005)^{10}} = \$136,686$$

at $8\% \rightarrow \$19,053$

at $10\% \rightarrow ?$

**Inflation vs. Interest:**

If inflation is 100% per year, then the $value (nominal cost) needs many $ = 1.005$

as if there is no interest rate.

So if we have an asset earning 100% per year, but inflation is 100% per year, the real value (after inflation) rises according to

$$An = \left(\frac{1 + i}{1 + f}\right) An$$

i.e., now $r = \frac{1 + i}{1 + f}$

The real interest rate is

$$r = \frac{1 + i}{1 + f} = \left(\frac{1 + f}{1 + i}\right)^{-1}$$

We have to use a reasonable approximation.

To account for inflation, we use the real interest rate

$= \text{nominal interest rate} - \text{inflation rate}$

**Now let $b = 0$**

Any time you have a periodic payment (i.e., or not) $b = 0$

- called an 'annuity'

In the simplest case, there is no initial payment

$(A_0 = PV = 0)$

so only periodic payments exist.

$$\Rightarrow A_n = b \left(\frac{(1+i)^n - 1}{i}\right)$$

Common kinds of questions:

1) Given $i, n, PMT$, how much is annuity worth in the future?

‘Future value of an annuity’

2) Given $i, n, PV$, find $PMT$

‘Sinking Fund’

3) Purchasing an annuity for reimbursement:

Fed just encourages retirees to purchase annuities

(= schedule of regular payments (out))

Here, $i$ is given (by bank)

one choses $PMT, n$

‘Present value of an annuity’

4) Amortization of a loan

You must make periodic payments to the bank.

In effect, the bank has purchased an annuity from you. (on their terms)