2nd order linear difference eq.

\[ y_{n+2} = y_{n+1} + y_n, \quad y_0, y_1 \]

Fibonacci sequence

\[ y_n = y_{n-1} + y_{n-2}, \quad y_0 = 0, y_1 = 1 \]

\[ y_2 = 2, y_3 = 3, y_4 = 5, y_5 = 8, 13, 21, \ldots \]

Q: What is \( y_n \)?

Recall we had some properties that apply to any 2nd linear difference eq.

- \( y_n^{(1)}, y_n^{(2)} \) solve a 2nd order LDE
  - \( c_1 y_n^{(1)} + c_2 y_n^{(2)} \) also

- \( y_n \) solves a hom LDE
- \( y_n \) any soln of corresponding LDE

\[ y_n^{(1)}, y_n^{(2)} \text{ is a solution of hom LDE} \]

To define a unique solution

- we now need 2 ICs: \( y_0, y_1 \)

(then we just make sure these 2
- guarantee the unique solution)

⇒ we need 2 arbitrary constants in our general solution
- values will be found by imposing the ICs.
- Does this always work?
- let's see!

Step 1: Find 2 solns of corresponding hom LDE

\[ y_n^{(1)}, y_n^{(2)} \]

Step 2: from linear combination

\[ c_1 y_n^{(1)} + c_2 y_n^{(2)} \]

Step 3: add \( y_0, y_1 \) of above LDE

\[ y_n^{(0)} \]

Step 4: \( y_n = c_1 y_n^{(1)} + c_2 y_n^{(2)} + y_0 \)

Can we always find \( c_1, c_2 \) s.t. ICs are satisfied?

\[ y_0^{(1)}, y_0^{(2)} \]

We hope \( y_n^{(1)}, y_n^{(2)} \) are a soln for any \( c_1, c_2 \)

from Theorem

We need

\[ y_0 + A = c_1 y_0^{(1)} + c_2 y_0^{(2)} \]
\[ y_1 + B = c_1 y_1^{(1)} + c_2 y_1^{(2)} \]

- linear system

\[ \begin{pmatrix} y_0 + A \\ y_1 + B \end{pmatrix} = \begin{pmatrix} c_1 y_0^{(1)} + c_2 y_0^{(2)} \\ c_1 y_1^{(1)} + c_2 y_1^{(2)} \end{pmatrix} \]

Now solve for \( c_1, c_2 \)

\[ \begin{pmatrix} y_0 + A \\ y_1 + B \end{pmatrix} = \begin{pmatrix} A - y_0^{(1)} \\ B - y_1^{(1)} \end{pmatrix} \]

This can be solved uniquely for \( c_1, c_2 \)

provided \( \det \begin{pmatrix} y_0 & y_0^{(1)} \\ y_1 & y_1^{(1)} \end{pmatrix} \neq 0 \)

\( A, y_0^{(1)}, y_0 \)
\( \Rightarrow A\frac{y_0^{(1)}}{y_0} + \frac{y_0}{y_1} \)

\( \Rightarrow \) provided the 2 solutions \( y_0^{(1)}, y_0^{(2)} \)
- are not multiples of each other!
- (since then \( \frac{y_1}{y_0} = k \))

[ \Rightarrow \phi_n, \phi_n are linearly independent ]

This can be generalized to \( n \)th order:

\[ y_n = c_1 y_n^{(1)} + c_2 y_n^{(2)} + \ldots + c_n y_n^{(n)} \]

where \( \sum y_n^{(n)} \) are linearly independent solutions.
So our strategy for 2nd order LDE:

Step 1: Find 2 sol's of hom LDE
(not multiples of each other)

Step 2: \( y'' = a y' + b y \)

Step 3: Add any solution \( y_n \) of non-hom LDE
\[ y_n = C_1 y_1 + C_2 y_2 + y_n \]
the general solution

Step 4: Apply IC's to solve for \( C_1, C_2 \)

We'll only look at hom.
Constant coefficient 2nd order LDE

\[ y'' + \alpha y' + \beta y = 0 \]

Since 1st order remain \( y_n = \lambda y \)
\[ \rightarrow \text{sol} \quad y_n = \lambda y \]
let's try such a solution \( y_n = \lambda y \)
\[ \lambda^{n+1} + \alpha \lambda^n + \beta \lambda^{n-1} = 0 \]
\[ \lambda^2 + \alpha \lambda + \beta = 0 \]

quadratic eqn for \( \lambda \) (\( \alpha, \beta \) given)
- characteristic equation

\[ \rightarrow 3 \text{ cases depending on} \]
- 2 real roots
- repeated roots
- complex roots

Case 1: real (distinct) roots
\[ \lambda^2 + \alpha \lambda + \beta = 0 \]
\[ \lambda = -\frac{\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2} \]
\[ \text{then} \quad y = C_1 \lambda_1 + C_2 \lambda_2 \quad \text{as} \quad \phi \neq k, \lambda_1 \neq \lambda_2 \]

Eq. 1: \( y_{1n} = -5y_n + 6y_n = 0 \)
\[ \lambda^2 - 5\lambda + 6 = 0 \]
\[ (\lambda - 3)(\lambda - 2) = 0 \]
\[ \lambda = 2, 3 \]
\[ y_n = C_1 \lambda_1 + C_2 \lambda_2 \]

Eq. 2: \( y_{2n} + 3y_n - 2y_n = 0 \)
\[ \lambda^2 + 3\lambda - 2 = 0 \]
\[ \lambda = 4, -1 \]
\[ y_n = C_1 4^2 + C_2 (-1)^2 \]

3. Fibonacci.
\[ y_{1n} - y_n - y_{-1} = 0 \]
\[ y_{1n} = \lambda^2 - 1 - 1 = 0 \]
\[ \lambda = 1 \pm \sqrt{1+4} = \frac{1 \pm \sqrt{5}}{2} \]
\[ y_n = C_1 \left( \frac{1 + \sqrt{5}}{2} \right)^n + C_2 \left( \frac{1 - \sqrt{5}}{2} \right)^n \]

with IC's: \( y_{10} = 1, y_{11} = 1 \)
\[ 1 = C_1 + C_2 \]
\[ 1 = C_1 \left( \frac{1 + \sqrt{5}}{2} \right) + C_2 \left( \frac{1 - \sqrt{5}}{2} \right) \]
\[ = \frac{1}{2} (C_1 + C_2) + \frac{\sqrt{5}}{2} (C_2 - C_1) \]
\[ 1 = \frac{\sqrt{5}}{2} (C_1 - C_2) + C_1 - C_2 = \frac{1}{\sqrt{5}} \]
\[ C_1 = \frac{1}{2} \left( \frac{1 + \sqrt{5}}{2} \right), C_2 = \frac{1}{2} \left( \frac{1 - \sqrt{5}}{2} \right) \]
\[ y_n = \frac{1 + \sqrt{5}}{2} \left( \frac{1 + \sqrt{5}}{2} \right)^n + \frac{1 - \sqrt{5}}{2} \left( \frac{1 - \sqrt{5}}{2} \right)^n \]
an integer!!
Since \( \left(1 - \frac{5}{2} \right) < 1 \), and hence \( a \) vanishes.

So for large \( n \),

\[
Y_n \approx \frac{1.618}{2} \cdot \left( \frac{1.618}{2} \right)^n
\]

So ratio of successive terms in Fibonacci sequence approaches \( \frac{\sqrt{5} + 1}{2} \approx 1.618 \).

The "golden ratio"

2 rectangles are similar.

Case II: Repeated roots

Only 1 root lies from \( \lambda \).

How to find a 2nd solution?

In this case, \( a^2 - 4a = 0 \) and \( \lambda = -\frac{\sqrt{5}}{2} \).

\[
\lambda = \frac{\sqrt{5}}{2}
\]

\[
Y_{n+1} + \lambda Y_n + \lambda^2 Y_{n-1} = 0
\]

\[
Y_{n+1} - 2\lambda Y_n + \lambda^2 Y_{n-1} = 0
\]

\[
Y_{n+2} - \lambda^2 Y_n = [\lambda^2 - \lambda^2] Y_n = 0
\]

So

\[
Y_{n+2} = \lambda Y_{n+1} - \lambda Y_n
\]

\[
Y_{n+3} = \lambda Y_{n+2} - \lambda Y_{n+1}
\]

So defining \( z_n = \lambda Y_{n+1} - \lambda Y_n \),

\[
z_{n+1} - z_n = 0 \implies z_n = c \lambda^n
\]

So \( Y_{n+2} = c \lambda^n \)

Order when

Since we only need \( a \) for \( \lambda^n \).

Let \( Y_0 = 0 \),

\[
Y_1 = c \lambda
\]

\[
Y_2 = \lambda (c \lambda) + c \lambda^2 = 2c \lambda^2
\]

\[
Y_3 = \lambda (2c \lambda^2) + c \lambda^3 = 3c \lambda^2
\]

\[
Y_n = c \lambda^n
\]

Prove by induction

2nd solution is \( c \lambda^n \)

and not a multiple of 1st since \( \frac{c \lambda^n}{c} = \frac{n}{2} \).

So

\[
Y_n = c_1 \lambda^n + c_2 n \lambda^n
\]

\[
\lambda = -\frac{\sqrt{5}}{2}
\]

Case III: Complex roots.

\( a^2 - 4a \implies \lambda = -\frac{\sqrt{5}}{2} \pm \frac{\sqrt{3}}{2} \sqrt{4 - \sqrt{5}} \lambda_1, \lambda_2 \)

\[
Y_n = c_1 \lambda_1^n + c_2 \lambda_2^n \quad \lambda_1, \lambda_2 \text{ complex}
\]

To solve, we want to express the (real) \( Y_n \)

in terms of real functions.

So we need to convert to real form.

\[
Y_n = Y_{n+1} - Y_{n-1}
\]

\[
Y_{n+1} - \lambda Y_n = 0
\]

\[
\lambda = \frac{\pm \sqrt{3}}{2}
\]

The easiest way to express powers of complex nos. is to use polar form.

Or complex exponential form.

Can now use to find modulus & argument.

In \( \lambda_1, \quad |\lambda| = \sqrt{(\text{Re}\lambda)^2 + (\text{Im}\lambda)^2}

\[
\text{Arg} \lambda = \tan^{-1} \frac{\text{Im}\lambda}{\text{Re}\lambda}
\]

So \( \lambda_1 = c e^{i \theta} \), \( \lambda_2 = -c e^{i \theta} \)

So solution can be written

\[
Y_n = c_1 \lambda_1^n + c_2 \lambda_2^n
\]

\[
= c_1 e^{-i \theta} + c_2 e^{i \theta}
\]

Since \( \lambda_1, \lambda_2 \) are complex.

When we solve for \( Y_0, Y_1 \),

\[
Y_n = c_1 e^{-i \theta} + c_2 e^{i \theta}
\]

Now use another property of\( \lambda_1, \lambda_2 \).
Theorem: If $y_n$ (complex) solves $x^2 + y_n + 2y_n = 0$, $y_n$ is real if

$$y_{n+1}^2 + y_n + 2y_n = 0$$

Then to do

$\text{Re}(y_n)$, $\text{Im}(y_n)$

Proof: $y_n = y_n^* + iy_n^*$

$\Rightarrow y_n = y_n^* + iy_n^*\Rightarrow y_n^* = \text{Re}(y_n)$

\[ y_{n+1} = 2y_n^* + 4y_n \]

now equate real & imaginary parts

$\therefore$ is a set

$\Rightarrow$ $\text{Re}(y_n)$ is a set

$\text{Im}(y_n)$ is a set

There are not multiples of each other

$\Rightarrow$ $x^2 + y_n = c_1 \cos y_n + c_4 \sin y_n$

$c_1, c_4$ real

We don't yet get anything new from $\text{Re}(e^{-in\theta})$, $\text{Im}(e^{-in\theta})$

Summary of Case III

If $\lambda$ is complex, $|\lambda| e^{i\theta}$

$\Rightarrow y_n = 1n^* \left( c_1 \cos n\theta + c_2 \sin n\theta \right)$

Exercise: Show $c_{80}$ is

$y_{n+1} - 2y_n + 4y_n = 0$

$\Rightarrow y_n = 2n \left( c_1 \cos \frac{n\theta}{2} + c_2 \sin \frac{n\theta}{2} \right)$