This week: aim to cover

- finish eigenvectors

- systems of ODEs

- homogeneous linear first order systems
Q6: how many eigenvectors can a matrix have?

Recall each eigenvector is really a **family of solutions**, all in the same direction.

By different eigenvectors, we mean **linearly independent vectors**, representing different directions.

\[
\text{A has } n \text{ distinct eigenvalues } \Rightarrow \text{A has } n \text{ linearly independent eigenvectors}
\]

Example
What about repeated eigenvalues?

Can have (e.g for 2 repeated eigenvalues):

1. 2 linearly independent eigenvectors
   ▷ Example

2. only 1 eigenvector ⇒ A is called defec-
   tive
   ▷ Example
What about complex eigenvalues?

For a real $A$, eigenvalues come in complex conjugate pairs

$$\lambda = p \pm iq$$

and so do eigenvectors

$$v = u \pm iw$$

Example
Systems of ODEs

Examples:

More general than single ODEs.

**Sometimes** can turn system into single ODE

Example

**Always** can turn single ODE into *system of 1st order ODEs*

Example
End of Lecture 27
Systems can be:

- linear/nonlinear
- 1st /2nd order (or higher)

We do only:

- 1st order constant coefficient systems
- (special) 2nd order constant coefficient systems
Linear systems of First order ODEs

General form \((2 \times 2)\)

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix}
= 
\begin{pmatrix}
a_{11}(t) & a_{12}(t) \\
a_{21}(t) & a_{22}(t)
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
+ 
\begin{pmatrix}
b_1(t) \\
b_2(t)
\end{pmatrix}
\]

or

\[
\dot{x} = A(t)x + b(t)
\]

We only do \textbf{constant coefficient systems}:

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix}
= 
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
+ 
\begin{pmatrix}
b_1(t) \\
b_2(t)
\end{pmatrix}
\]

\[
\dot{x} = Ax + b(t)
\]

The basic theory is the same, but it’s harder to find analytical solutions for case of \(A(t)\)
First order linear systems: Theory

Similar to 2nd order linear ODEs:

**Theorem:** Let $x_1, x_2$ be 2 linearly independent solutions of a homogenous $2 \times 2$ linear system of 1st order DEs. Then

$$x = c_1 x_1 + c_2 x_2$$

is the general solution of the ODE.

$\Rightarrow$ every solution is of this form!

Note: 2 arbitrary constants

$3 \times 3$ linear system $\rightarrow$ 3 linearly independent solutions $\rightarrow$ 3 arbitrary constants
**Theorem:** Let $x_c$ be the general solution of a homogenous $2 \times 2$ linear system of 1st order DEs

$$\dot{x} = A(t)x$$

and $x_p$ be a **particular solution** of the inhomogeneous system:

$$\dot{x} = A(t)x + b(t)$$

then the general solution of the inhomogeneous system is

$$x = x_c + x_p = c_1x_1 + c_2x_2 + x_p$$

→ need to be able to solve homogeneous systems first!
Homogeneous constant coefficient 1st order systems

\[ \dot{x} = Ax \]

Like 2nd order ODEs, look for exponential solutions:

try \( x = e^{\lambda t}v \) (\( v \) a constant vector)

\[ \rightarrow \]

\[ \lambda e^{\lambda t}v = Ae^{\lambda t}v \]

\[ \Rightarrow \]

\[ Av = \lambda v \]

the eigenvalue equation!

\[ \rightarrow \] we know how to find \( \lambda, v \)
Like 2nd order ODEs, form of solution depends on whether eigenvalues are:

1. distinct

2. complex

3. repeated
Distinct eigenvalues

A has \( n \) distinct eigenvalues \( \Rightarrow \) A has \( n \) linearly independent eigenvectors

\[ \text{→ system has } n \text{ linearly independent solutions, each of the form } x_i = e^{\lambda_i t} v_i \]

\[ \text{→ General solution } \]

\[ x = \sum_{i=1}^{n} c_i x_i = \sum_{i=1}^{n} c_i e^{\lambda_i t} v_i \]

\[ \triangle \text{ Example} \]
**Complex eigenvalues**

For a real $A$, eigenvalues come in complex conjugate pairs

$$\lambda = p \pm iq$$

and so do eigenvectors

$$v = u \pm iw$$

but **usually we want real-valued solutions!!**

⇒ take real/imaginary parts of 1 complex solution → 2 linearly independent real solutions

$$x_1 = \Re e^{(p+iq)t}(u+iw) = e^{pt}(\cos qtu - \sin qtw)$$

$$x_2 = \Im e^{(p+iq)t}(u+iw) = e^{pt}(\cos qtw + \sin qtu)$$

Example
End of Lecture 28
Repeated eigenvalues: full set of eigenvectors

If A has a full set of eigenvectors, then we have a full set of solutions:

\[ x_i = e^{\lambda t} v_i \]

Example
Repeated eigenvalues: defective matrix

If we have \( < n \) eigenvectors, not every solution has the form \( e^{\lambda t}v \)

\( \Rightarrow \) try ‘Variation of Parameters’

look for solution \( x_2 = x_1w(t) = e^{\lambda t}vw(t) \) ?

or \( x_2 = tx_1 \) ?

we’ll try \( x_2 = e^{\lambda t}w(t) \)

\( \Rightarrow \)

\[ \lambda e^{\lambda t}w(t) + e^{\lambda t}\dot{w}(t) = Ae^{\lambda t}w(t) \]

\[ \lambda w(t) + \dot{w}(t) = Aw(t) \]

\[ \dot{w}(t) = (A - \lambda I)w(t) \]
try \( w(t) = ta + b \)

\[
\rightarrow
\]

\[
a = (A - \lambda I)(ta + b) = t(A - \lambda I)a + (A - \lambda I)b
\]

\[
\Rightarrow (A - \lambda I)a = 0 \Rightarrow a = v
\]

and \((A - \lambda I)b = a = v\)

if the \(2 \times 2\) matrix \(A\) has only 1 eigenvector \(v\), one solution is \(x_1 = e^{\lambda t}v\). The second is

\[
x_2 = e^{\lambda t}(tv + b)
\]

where \((A - \lambda I)b = v\)

Note: \((A - \lambda I)^2b = 0\) but \((A - \lambda I)b \neq 0\)

\(<\text{ Example}>\)
End of Week 10

Notes §9.1–9.3

Now do Sheet 10