This week: aim to cover

- Second order linear ODEs
- ‘Variation of parameters’
- Homogenous constant coefficient Second order ODEs
Second order ODEs

Important class of DEs — mechanics

**Newton's Law of Motion**

\[ ma = F \]

\[ \ddot{x} = F(t, x, \dot{x})/m \]

in 1 dimension \(\rightarrow\) a second order ODE

In general \(\rightarrow\) system of second order ODEs

We do only ....
Second order Linear ODEs

\[ y'' + P(x)y' + Q(x)y = R(x) \]

Again, \( P, Q, R \) cts \( \Rightarrow \) unique solution to IVP with ICs

\[ y(x_0) = \alpha, y'(x_0) = \beta \]

If \( R \equiv 0 \), DE is homogeneous.

Linearity \( \Rightarrow \) can say a lot about form of general solution

\( \rightarrow \) strategy for finding solutions
**Theorem:** Let $y_1, y_2$ be 2 solutions of a homogenous linear 2nd order DE. Then

$$y = c_1y_1 + c_2y_2$$

— a **linear combination** — is also a solution.

**Proof:** see Notes.

Relies on DE being both **linear** and **homogeneous**

BUT is **every** solution of this form?

NO
**Definition:** 2 functions are **linearly independent** over \([a, b]\) if they are not multiples of each other.

**Theorem:** Let \(y_1, y_2\) be 2 **linearly independent solutions** of a homogenous linear 2nd order DE. Then

\[ y = c_1y_1 + c_2y_2 \]

is the **general solution** of the ODE.

⇒ *every solution is of this form!*

Note: 2 arbitrary constants

What about inhomogeneous ODE?
Theorem: Let \( y \) be any solution of linear 2nd order DE \( y'' + P(x)y' + Q(x)y = R(x) \)

\( y_p \) be a particular solution of same DE \( y'' + P(x)y' + Q(x)y = R(x) \)

\( y_c \) be any solution of corresponding homogeneous DE \( y'' + P(x)y' + Q(x)y = 0 \)

then

\[ y = y_c + y_p = c_1y_1 + c_2y_2 + y_p \]

Proof: see Notes

The part \( y_c \) is called the complementary function

\( \Rightarrow \) general solution = general solution of homogeneous DE + particular solution of inhom. DE
Strategy for linear 2nd order DEs

⇒ need to be able to:

1. find solutions $y_1, y_2$ of homogeneous DE

2. find a particular solution $y_p$ of inhomogeneous DE

then we’re done!

but first ...

a method to find $y_2$ (and $y_p$) if we know $y_1$
Variation of parameters / reduction of order

Given $y_1$, we can get the rest of the solution!

Method: change dependent variable

— look for a solution of the form $y = y_1(x)v(x)$ where $y_1$ solves the (homogeneous) DE

$\rightarrow$ 1st order linear DE for $v'(x)$

(if not, you’ve made a mistake)
Let \( y = y_1v \)

\[ y' = y_1'v + y_1v', \quad y'' = y_1''v + 2y_1'v' + y_1v'' \]

sub. into DE:

\[ y''_1v + 2y_1'v' + y_1v'' + P(y_1'v + y_1v') + Qy_1v = R \]

write as DE for \( v \)

\[ y_1''v + (2y_1' + Py_1)v' + (y_1'' + Py_1 + Qy_1)v = R \]

but

\[ y_1'' + P(x)y_1' + Q(x)y_1 = 0 \]

\((y_1 \text{ solves hom. DE}) \rightarrow \)

\[ y_1v'' + (2y_1' + Py_1)v' = R \]

1st order linear DE for \( v' \)

\( \triangle \) Example
End of Lecture 19?
How to find $y_1$?

in 620143, we do the simplest class:

**Constant coefficient homogeneous DEs**

$$ay'' + by' + cy = 0$$

$a, b, c$ constant

Can we find a function whose 2nd deriv. is a linear comb. of its deriv. and itself?

Can we find a function whose deriv. is a multiple of itself?

YES

$$\frac{d}{dx}e^{kx} = ke^{kx}$$

→ let's look for solutions that are exponential
Try \( y = e^{\lambda x} \) in DE:

\[
a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + c e^{\lambda x} = 0
\]

but \( e^{\lambda x} \neq 0 \), so

\[
a\lambda^2 + b\lambda + c = 0
\]

Roots of this characteristic equation \( \rightarrow \) solutions \( y_1, y_2 \) of the homogeneous DE

Like any quadratic, 3 cases:

1. 2 distinct real roots \( b^2 - 4ac > 0 \)

2. 2 repeated roots \( b^2 - 4ac = 0 \)

3. 2 complex conjugate roots \( b^2 - 4ac < 0 \)
Case 1: 2 distinct real roots

We have 2 values $\lambda_1, \lambda_2$

→ 2 linearly independent solutions

\[
y_1 = e^{\lambda_1 x}
\]

\[
y_2 = e^{\lambda_2 x}
\]

→ general solution

\[
y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}
\]

Example
Case 2: repeated roots

We have 1 value $\lambda = -b/2a$

→ 1 solution only!!

$$y_1 = e^{\lambda x}$$

Use variation of parameters to get 2nd solution $y_2$:

let $y(x) = y_1(x)v(x) = e^{\lambda x}v(x)$

→ $y' = \lambda e^{\lambda x}v(x) + e^{\lambda x}v'(x)$,

$y'' = \lambda^2 e^{\lambda x}v(x) + 2\lambda e^{\lambda x}v'(x) + e^{\lambda x}v''(x)$

→

$$a[\lambda^2 e^{\lambda x}v(x) + 2\lambda e^{\lambda x}v'(x) + e^{\lambda x}v''(x)] + b[\lambda e^{\lambda x}v(x) + e^{\lambda x}v'(x)] + ce^{\lambda x}v(x) = 0$$
\[ ae^{\lambda x}v'' + [2a\lambda + b]e^{\lambda x}v'(x) = 0 \]
since term in \( v \) must vanish!

but \( \lambda = -b/2a \Rightarrow [2a\lambda + b] = 0! \) so ...

\[ v'' = 0 \]

\[ \Rightarrow \]

\[ v(x) = c_1 x + c_2 \]

\[ \Rightarrow \]

\[ y_2(x) = y_1(x)v(x) = e^{\lambda x}(c_1 x + c_2) \]

the part \( c_2 e^{\lambda x} \) we knew before; but the part \( c_1 x e^{\lambda x} \) is the 2nd solution

\[ \text{The constant coefft. homogeneous DE with repeated roots of the characteristic equation has 2 solutions} \]

\[ y_1(x) = e^{\lambda x}; \quad y_2(x) = x e^{\lambda x} \]
Since $y_1/y_2 \neq \text{constant}$, the solutions are linearly independent

→ general solution

$$y(x) = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$$

Example
End of Week 7

Notes §6.1–6.4

Now do Sheet 7

Next week: Ass. Prof. Landman!