Assignment 2: Boundary value problems
Due: Monday, April 28

Note: Submit copies of computer programs (Matlab/Octave) and sufficient relevant output (use the diary function).

You can email me assignments if you want.

1 Compare the methods

Solve the BVP, depending on a parameter $p$

$$y'' + 3py/(p + x^2)^2 = 0$$

subject to $y(\pm 0.1) = \pm 0.1/\sqrt{p + 0.01}$. The exact solution is $y(x) = x/\sqrt{p + x^2}$.

Consider both the cases $p = 1$ (easy) and $p = 10^{-4}$ (hard).

Solve this problem:

a. by second order central finite differences and uniform spacing. Write the code yourself as an M-file. The file `fd.m` is probably a good place to start, though the matrices are not assembled in an efficient way. You should always have tridiagonal matrices appearing. For quick ways to generate these, see `help diag`.

b. by finite elements. Adapt the code `ellfemdrv` from Lab 5. You will have to write function M-files for each coefficient function (3 are trivial) instead of the `sin(x)` etc. currently used for illustration.

c. by Chebyshev collocation. Adapt the code `p13.m` from Lab 5. To do this, you have to change things in a few ways:

- rewrite the BVP in terms of a new independent variable so the interval is from [-1,1].
- the line in `p13.m`, `D2 = D2(2:N,2:N);` removes the first and last row and column from the differentiation matrix because the Dirichlet boundary conditions are homogeneous. In this case, the matrix will have `D2` plus a diagonal matrix for the interior nodes, plus the first and last rows are retained to enforce the boundary conditions. See me for more details.

d. by shooting. Adapt your code from Lab 5.

In each case, compute and plot the error (either the $1$-norm or $\infty$-norm will do) as $N$, the number of grid points (or basis functions) is increased. Discuss. I suspect this problem will be tough for Chebyshev collocation.
2 The laminar boundary layer

The Falkner-Skan equations describing a similarity solution for the flow past a flat plate with an external flow of the form \( u_e = U x^\beta \) are

\[
f''' + ff'' + \beta(1 - (f')^2) = 0
\]

subject to

\[
f(0) = f'(0) = 0, f'(\eta) \to 1 \text{ as } \eta \to \infty
\]

Here \( \eta \) is a similarity variable and the streamwise velocity is related to \( f'(\eta) \).

This problem is posed on a semi-infinite interval. One approach is to solve for a finite range \([0,R]\) for ‘sufficiently’ large \( R \). Here, try \( R = 4, 5, 6 \).

Solve this nonlinear problem by shooting (from 0). For the case \( \beta = 0.5 \), what value of \( f''(0) \) do you find? Plot your solutions \((f, f', f''\) on a single plot) for various values of \( \beta \), including \( \beta = 0 \) (Blasius solution).

Verify that ‘physically realistic’ solutions — meaning solutions where \( f' \) is positive and approaches 1 from below for large \( \eta \) — exist for \(-0.19884 \leq \beta \leq 2\).