Lab sheet 6. More linear systems

1 Fitting a line....

Run the demo ShowLSFit.m to see two linear fits to the square root function. With 2 points, it is just the linear interpolant. With 100 points, we have a fitting problem. Note the use of the Matlab backslash command - if the matrix is rectangular, Matlab automatically solves the least squares problem (using QR factorization). I’m not sure if Matlab uses Householder transformations or not.

2 Normal equations

Use Matlab to Choleski factorize the normal equations arising from the matrix

\[
A = \begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 - \eta
\end{bmatrix}
\]

where \( \eta = 10^{-10} \). Examine \( A^T A \) and explain what’s going on. Experiment for other values of \( \eta \).

What happens if you QR factorize instead?

3 Normal equations and conditioning

Generate some matrices (not necessarily square) e.g. using \texttt{rand}, \texttt{hilb}, \texttt{pascal} etc. Find their condition numbers (use \texttt{cond}), form \( A^T A \) and check its condition number.

4 See QR

The M-file ShowQR.m illustrates how QR factorization works to produce a triangular matrix R. The actual code given uses a different procedure called Givens rotations rather than Householder transformations but the idea is the same.

5 Conditioning of least squares problems

The sensitivity of least squares problems is rather subtle but can roughly be stated as:

\[
\frac{\| \hat{x} - x \|_2}{\| x \|_2} \approx \epsilon_{mach} (\text{cond}_2(A) + \| Ax - b \|_2 \text{cond}_2(A)^2)
\]

This illustrated in the scriptfile ShowLSq.m - try the values \( m = 10, n = 4, \text{cond} = 10^7 \).
6 Row operations are not for least squares

Use Matlab’s forward slash to solve the overdetermined system

\[
\begin{bmatrix}
1 & 0 \\
2 & 1 \\
3 & 0
\end{bmatrix}
\begin{bmatrix}
x
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
3 \\
5
\end{bmatrix}
\]

and find the residual (in the 2-norm). Now by hand reduce the system to reduced row echelon form (i.e. using row operations ) and solve the resulting overdetermined system. Compare the answers and residuals with your first answers. This shows that row operations do not preserve the solution of a least squares problem.

7 What’s so good about orthogonal matrices?

Show (mathematically) that

a. the 2-norm of the residual of an overdetermined system is unchanged by an orthogonal transformation

b. the Householder matrix \( P = I - 2 \frac{vv^T}{v^Tv} \) where \( v \) is any vector , is symmetric and orthogonal

8 Semantics

Explain the quote (Trefethen and Bau, Numerical Linear Algebra):

Gram-Schmidt is triangular orthogonalization; QR is orthogonal triangularization.