THREE KNESER LEMMATA

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1. Introduction

This note contains three versions of a result which is sometimes referred to as Kneser’s lemma and states that (with respect to a fixed triangulation) there is a bound on the number of disjointly embedded normal surfaces in a 3–manifold which satisfy an extra non–triviality condition. One often stipulates that no two of the surfaces be parallel in $M$. We relax this condition and merely assume that no two of the surfaces are normal isotopic, under which we understand that they do not have the same normal surface coordinates.

The three versions address in turn compact, closed and cusped 3–manifolds (where we restrict ourselves to torus cusps). Techniques are borrowed from [4, 1] and lectures by Hyam Rubinstein. We also extend the main result of [1] to show that if $T$ is the number of tetrahedra in a triangulation of a closed, irreducible 3–manifold $M$, then if a 2–sided, geometrically incompressible surface in a closed, irreducible 3–manifold $M$ has more than or equal to $\frac{3}{2}T$ components, then two of them must be parallel.

2. Preliminaries

We will assume familiarity with normal surface theory in both the compact and the cusped setting, and only emphasise that a (ideal) triangulation of $M$ is a decomposition of $M$ into the union of (ideal) tetrahedra identified along faces, where the identification of a face to itself is not allowed. This is sometimes called a pseudo–triangulation, and includes triangulations in the traditional sense. We will often drop the adjective ideal. In triangulations of closed and cusped 3–manifolds, there are no faces of tetrahedra which are not paired with some other face. In a triangulation of a compact 3–manifold, the link of a vertex is either a disc or a sphere, and in a triangulation of a 3–manifold with torus cusps, the link of a vertex

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is a torus. Moreover, a connected normal surface which is only made up of triangles is a vertex linking surface.

We now consider face identifications involving a single tetrahedron. There are six ways to identify two faces of a tetrahedron, three are ”orientable” and three are ”non–orientable" (see [2, 3]). If the gluing is non–orientable, then either an edge is identified to itself with opposite orientation, or a Möbius band is a subsurface in a vertex linking surface, which therefore must be non–orientable. This shows that these face pairings cannot occur in any triangulation of any compact manifold or any manifold with torus cusps.

The orientable pairings can all occur, and result in identification spaces whose interior is either homeomorphic to a 3–ball or a solid torus. We will call these cases the ball pairing and the torus pairing. Thus, there are two compact manifolds with boundary which admit a 1–tetrahedron triangulation. Analysing the possible identifications for the remaining two faces, one finds that there are only four possible combinations, all of which yield spherical vertex links. The corresponding closed 3–manifolds are \( S^3 \) and the lense spaces \( L(4, 1) \) and \( L(5, 2) \) (see [2]).

3. Compact manifolds

**Lemma 1.** Let \( M \) be a compact 3–manifold with triangulation \( T \), and \( S \) be an embedded normal surface such that no two components of \( S \) are normal isotopic and no component is a vertex linking surface. Then the number \( |S| \) of the components of \( S \) satisfies:

\[
|S| \leq 3T + \frac{3}{2} \dim H_2(M, \partial M; \mathbb{Z}_2),
\]

where \( T \) is the number of tetrahedra in \( T \). Moreover, if \( S \) is a 2–sided surface, then \( |S| \leq 3T + \frac{1}{2} \dim H_2(M, \partial M; \mathbb{Z}_2) \).

**Proof.** Since no component of \( S \) is a vertex linking surface, the disjoint union of \( S \) with the vertex linking surfaces contains no normal isotopic pair of surfaces. Denote this union by \( S \) again. The normal discs of \( S \) divide a tetrahedron of \( T \) into the following types of regions:

1. **slabs**: trivial \( I \)–bundles over normal discs,
2. **thick regions**: truncated tetrahedra and truncated triangular prisms,
3. **vertex regions**: ”small” tetrahedra which contain a vertex of \( T \).

In total, there are at most \( V + 2T \) components of \( M - S \) which contain at least one of the latter types of regions – the remaining components are entirely made up
of slabs. Each such slab component is either a trivial or a twisted $I$–bundle over a normal surface. If a slab component is a trivial $I$–bundle, then there are two cases. Either its two boundary components are normal isotopic components of $S$, which contradicts our assumption. Or a 2–sided surface in $S$ is the boundary of a twisted $I$–bundle over a 1–sided surface in $S$.

All other slab components are twisted $I$–bundles over a normal surface not in $S$. Since this core surface is not normal isotopic to any of the components of $S$, we may add it disjointly to $S$. Now let $F$ be a 1–sided component of $M$. Then a small regular neighbourhood of $F$ is a twisted $I$–bundle with boundary an embedded 2–sided surface $N$ in $M$. If $N$ is not normal isotopic to a component of $S$, we join it to $S$.

Let $\{V_i\}$ be the set of all vertex linking surfaces in $S$, $\{F_i\}$ be the set of 1–sided surfaces, the set of corresponding 2–sided surfaces be $\{N_i\}$, and all remaining 2–sided surfaces be in the set $\{S_i\}$. The first set contains $V$ elements, and the second and third have the same magnitude. The fourth set contains 2–sided separating and non–separating surfaces; some of which may be parallel or parallel to some surfaces in $\{N_i\}$, but not normal isotopic.

Each $F_i$ is a 1–sided and therefore non–separating surface in $M$, and it is not parallel to any other component of $S$, since a 1–sided surface cannot be pushed off itself. None of these surfaces bounds a 3–dimensional submanifold of $M$, and hence each determines a non–zero element of $H_2(M, \partial M; \mathbb{Z}_2)$. Moreover, no subset of these surfaces bounds, and hence they are linearly independent elements of $H_2(M, \partial M; \mathbb{Z}_2)$. Put $R = \dim H_2(M, \partial M; \mathbb{Z}_2)$.

Assume that there is a 2–sided component $F$ of $S$ which is not a vertex linking surface and meets no thick region. Then $F$ bounds an $I$–bundle to either side. If one of them is trivial, then there is a component in $S$ which is normal isotopic to $F$. Hence both of these $I$–bundles are twisted. But then $M$ is decomposed into two twisted $I$–bundles glued along their boundaries, and $S = F$ does not meet any thick region. This is not possible. Thus, each element of $\{N_i\}$ and $\{S_i\}$ meets at least one thick region. Also note that each $N_i$ meets thick regions only on one side, and that each $S_i$ meets at least one thick region on each side. We now use a doubling trick from [1]: push each $N_i$ off itself away from the twisted $I$–bundle, and call the resulting copy $N_i'$; and push each $S_i$ off itself and call the resulting copy $S_i'$. This can be done such that all surfaces are still disjoint. There are at most $6T$ normal discs in the boundaries of thick regions, and (following the literature) we will call them bad. Each of the surfaces in the sets $\{N_i'\}$, $\{S_i\}$, $\{S_i''\}$ meets at least one thick
region in at least one bad disc. Thus, $2|S| = 2|\{V_i\}| + 2|\{F_i\}| + 2|\{N_i\}| + 2|\{S_i\}| = 2|\{V_i\}| + 3|\{F_i\}| + |\{N_i\}| + 2|\{S_i\}| \leq 2V + 3R + 6T$. Dividing by two and subtracting the vertex linking surfaces gives the desired inequality. 

4. Closed manifolds

For a closed 3–manifold, we can improve the above bound as follows.

**Lemma 2.** Let $M$ be a closed 3–manifold with triangulation $T$. Let $S$ be an embedded normal surface such that no two components of $S$ are normal isotopic and no component is a vertex linking surface. Let $|S|$ be the number of components of $S$, $T$ be the number of tetrahedra and $E_i$ be the number of edges of degree $i$ in $T$. Then

$$|S| \leq T + \frac{3}{2} \dim H_2(M, \mathbb{Z}_2) + E_1 + E_2. \tag{2}$$

If all components of $S$ are 2–sided, then $|S| \leq T + \frac{1}{2} \dim H_2(M, \mathbb{Z}_2) + E_1 + E_2$.

**Proof.** As in the previous proof, we disjointly add all vertex linking surfaces to $S$. Assume that there is a component $F$ of $S$ which meets a tetrahedron containing a degree one edge in a quadrilateral. There is only one quadrilateral type that $F$ can meet such a tetrahedron in. A normal disc of this type forms an annulus around the degree–one edge, and we see a compression disc for $F$. We can perform a disc exchange, where we remove the quadrilateral, and glue two triangles to the resulting holes in the surface. We obtain at most two components after this operation, and they could be normal isotopic to other components of $S$. However, if we now successively perform disc exchanges deleting all quadrilateral discs in the tetrahedron, this does not introduce any other pairwise normal isotopic surfaces. Denote by $S'$ the surface obtained from $S$ by replacing all quadrilaterals by triangles in all tetrahedra containing degree one edges, and deleting normal isotopic ”doubles”. Then $|S| \leq |S'| + E$, and $S'$ meets tetrahedra with degree one edges only in triangles.

We now continue the proof of the first lemma, where we replace $H_2(M, \partial M; \mathbb{Z}_2)$ by $H_2(M, \mathbb{Z}_2)$ and $S$ by $S'$ throughout, and delete the last two sentences.

We know that each of the surfaces in the sets $\{N'_i\}$, $\{S_i\}$, $\{S'_i\}$ meets at least one thick region in at least one bad disc, and we claim that they contain at least three bad discs unless the thick region contains a degree–two edge.

Let $F$ be one of the above surfaces. Then there is a thick region $r$ which it meets in a bad disc, and we first prove that $r$ may be chosen to be a truncated prism.
By way of contradiction, assume that all bad discs in $F$ are triangles contained in truncated tetrahedra. Choose $r'$ to be a truncated tetrahedra, containing a bad disc $\Delta$ of $F$. Then $r'$ is identified along the hexagonal faces containing $\Delta$ to other truncated tetrahedra, and it follows that $F$ must be a vertex linking surface, contradicting our choice. Hence we may assume that $r$ is a truncated prism. If it is glued along its two hexagonal boundary faces to two other thick regions, then $F$ contains at least three bad discs. If these two faces are glued to each other, then as discussed in Section 2, there are precisely two cases to consider. However, due to our choice of $S'$, the ball pairing cannot occur. Hence, we have a torus pairing. Here, the three bad discs are glued together to form an annulus, and $F$ contains at least three bad discs. Hence assume that $r$ meets exactly one other thick region $r'$ along its hexagonal faces. If $r'$ is a truncated tetrahedron, then it is not hard to see that $F$ must contain at least three bad discs. Hence assume that $r'$ is a truncated prism. Then $r$ and $r'$ meeting along one hexagon implies that $F$ contains at least two bad discs. Identification along the other hexagon yields that $F$ contains either all six bad discs in the boundaries of $r$ and $r'$ or exactly two. The latter is only possible if the edge of $r$ not meeting the quadrilaterals is of degree two.

Thus, each of the surfaces in the sets $\{N_i\}', \{S_i\}, \{S_i'\}$ contains at least two bad discs, and if they do not meet regions containing degree–two edges, they contain at least three bad discs. Their number is thus bounded above by $2T + E_2$. We now compute twice the number of components: $2|S| \leq 2|S'| + 2E \leq 4|\{N_i\}| + 2|\{S_i\}| \leq 3R + 2T + 2E_1 + E_2$. Dividing by two gives the desired result.

**Remark 3.** One often looks for interesting normal surfaces in prime manifolds. It is shown in [2] that if a minimal triangulation with $\geq 3$ tetrahedra of a closed prime 3–manifold has an edge of degree one or two, then the underlying manifold is non–orientable and contains an embedded 2–sided projective plane.

**Remark 4.** The above bound can also be obtained for closed normal surfaces in compact manifolds; where the quantities $E_i$ only concern non–boundary edges.

The following result is often referred to as Knese–Haken finiteness, and we remove the condition that $M$ be orientable from the main result of [1], and improve the bound $\leq 2T$ given there:

**Corollary 5.** Let $M$ be a closed, irreducible 3–manifold with triangulation $T$. Let $S$ be an embedded, 2–sided, geometrically incompressible surface such that no two
components of $S$ are parallel. Then the number $|S|$ of components of $S$ satisfies:

\[(3) \quad |S| \leq T + \frac{1}{2} \dim H_2(M, \mathbb{Z}_2) \leq \frac{3}{2} T < 2T,\]

where $T$ is the number of tetrahedra in $T$.

Proof. Since $M$ is irreducible and $S$ is geometrically incompressible, we may use isotopies to arrange that $S$ is normal. Moreover, under these assumptions, we may isotope $S$ to meet tetrahedra containing edges of degree one or two only in triangles when the offending situations of the previous proof occur. We thus arrive at the first inequality. Jaco’s bound from [1], that $\dim H_2(M, \mathbb{Z}_2) = \dim H_1(M, \mathbb{Z}_2) \leq T + 1$, completes the proof.

\[ \blacksquare \]

5. Cusped manifolds

Lemma 6. Let $M$ be the interior of a compact 3–manifold $\overline{M}$ with boundary consisting of a non–empty disjoint union of tori. Let $T$ be an ideal triangulation of $M$ and $S$ be an embedded normal surface such that no two components of $S$ are normal isotopic and no component is a vertex linking surface. Let $|S|$ be the number of components of $S$, $T$ be the number of tetrahedra and $E_i$ be the number of edges of degree $i$ in $T$. Then

\[(4) \quad |S| \leq T + \frac{3}{2} \dim H_2(\overline{M}, \partial \overline{M}; \mathbb{Z}_2) + E_1 + E_2.\]

Proof. The proof is based on the proof for closed manifolds, and we only point out the main differences. The surface $S$ meets each tetrahedron in at most finitely many quadrilaterals, but possibly infinitely many triangles. If there are infinitely many triangles near an ideal vertex $v$ of $T$ in one tetrahedron, then there are infinitely many triangles near this vertex in all tetrahedra containing it. We therefore find no region associated to $v$. Thus, we only add the vertex linking surfaces to $S$ which are near the cusps of $M$ that have no cusp of $S$ embedded into.

We now use the definitions for regions from earlier, where vertices are now understood to be ideal. There are finitely many thick and vertex regions, and finitely many slab regions involving quads, but possibly infinitely many slab regions involving triangles. Similar to the previous proofs, the number of 1–sided surfaces in $S$ is bounded above by $\dim H_2(\overline{M}, \partial \overline{M}; \mathbb{Z}_2)$. This can be seen by choosing a compact core $M^c$ of $M$ with the property that the interior of $F \cap M^c$ is homeomorphic to $F$ for each component $F$ of $S$. As before, we then only need to count 2–sided
components of $S$ which meet at least one thick region. This can be done analogous to the closed case.

\textbf{Remark 7.} It is shown in [3] that a minimal, ideal hyperbolic triangulation of a complete hyperbolic 3–manifold with torus cusps contains no edges of degree one, two or three.

\textbf{References}


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