HOMOLOGICAL STABILITY FOR SYMMETRIC COMPLEMENTS

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Abstract. Conjecture F from [VW12] states that the complements of closures of certain strata of the symmetric power of a smooth irreducible complex variety exhibit rational homological stability. We prove a generalization of this conjecture to the case of connected manifolds of dimension at least 2 and give an explicit homological stability range.

1. Introduction

The goal of this paper is to prove a generalization of a conjecture of Vakil and Wood (Conjecture F of [VW12]). This conjecture concerns homological stability for certain subspaces of the symmetric powers defined as the complements of closures of certain strata. Here a sequence of spaces $X_k$ is said to have homological stability if the homology groups $H_i(X_k)$ are independent of $k$ for $k \gg i$.

We begin by defining the relevant subspaces of symmetric powers. Let $\text{Sym}_k(M)$ denote the symmetric power $M^k/\mathfrak{S}_k$ of a space $M$. Here $\mathfrak{S}_k$ denotes the symmetric group on $k$ letters acting by permuting the terms. To any element of $\text{Sym}_k(M)$ we can associate a way of writing the number $k$ as a sum of positive integers by counting with what multiplicity each point appears. Such a sum is called a partition of $k$. For example, to the element $\{m_1, m_2, m_3\} \in \text{Sym}_4(M)$ with $m_1 \neq m_2 \neq m_3$ we can associate the partition $4 = 1 + 1 + 2$. Using this one can define the following subspaces of $\text{Sym}_k(M)$.

Definition 1.1. Let $\lambda$ be a partition of $k$.

(i) Let $S_\lambda(M)$ be the subspace of $\text{Sym}_k(M)$ consisting of elements that have associated partition equal to $\lambda$. We call this the stratum corresponding to $\lambda$.

(ii) Let $W_\lambda(M)$ be the complement of the closure of $S_\lambda(M)$ in $\text{Sym}_k(M)$. We call this the symmetric complement associated to $\lambda$.

A point in $S_\lambda(M)$ can be viewed as a configuration of distinct points labeled by natural numbers. For $\lambda = 4 + 4 + 5$ for example, $S_\lambda(M)$ is the configuration space of 3 distinct points in $M$, two of which are labeled by the number 4 and with the other point labeled by the number 5. The two points labeled by the number 4 are indistinguishable from each other but can be distinguished from the point labeled by the number 5. If $\lambda = 1 + \cdots + 1$ is a partition of $k$, then $S_\lambda(M)$ is the configuration space of $k$ distinct unordered points in $M$ often denoted $C_k(M)$. In general these spaces are homeomorphic to the colored configuration spaces considered in [Chu12].

One can think of $W_\lambda(M)$ as those elements of $\text{Sym}_k(M)$ that cannot be made to have associated partition $\lambda$ by an arbitrarily small perturbation. For example, an element of $\text{Sym}_r(M)$ is in $W_\lambda(M)$ with $\lambda = 1 + 1 + 1 + 2 + 2$ if all but possibly one point have multiplicity four or higher. Partitions of $k$ of the form $1 + \cdots + 1 + (c+1)$ yield the spaces known as the bounded symmetric powers which are often denoted $\text{Sym}_k^c(M)$. The bounded symmetric power of $M$ can be defined as the subspace of $\text{Sym}_k(M)$ where no point of $M$ has multiplicity greater than $c$. For $c = 1$, this is simply $C_k(M)$ the configuration space of $k$ distinct unordered particles in $M$.

Conjecture F pertains to the limiting behavior of symmetric complements as the number of particles increases. To state it we need the following construction. If $\lambda$ is a partition of $k$ we can obtain from it a partition of $k + 1$ by adding another 1 to $\lambda$. More generally we can add $j$ additional 1’s to obtain
a partition of \( k + j \), which we denote by \( 1^j \lambda \). In other words, if \( \lambda = m_1 + \cdots + m_i \), then \( 1^j \lambda \) is the partition \( 1 + \cdots + 1 + m_1 + \cdots + m_i \) where there are \( j \) additional 1’s. In \([VW12]\), Vakil and Wood made the following conjecture.

**Conjecture 1.2** (Conjecture F). For any irreducible smooth complex variety \( X \), \( \dim H_i(W_{1; \lambda}(X); \mathbb{Q}) = \dim H_i(W_{1; \lambda+1}(X); \mathbb{Q}) \) for \( j \gg i \).

We prove this conjecture, generalize it to the case of arbitrary connected manifolds of dimension at least 2 and give an explicit homological stability range. That is, we prove the following theorem.

**Theorem 1.3.** Let \( M \) be a connected manifold of dimension \( d \geq 2 \). We have that

\[
H_i(W_{1; \lambda}(M); \mathbb{Q}) \cong H_i(W_{1; \lambda+1}(M); \mathbb{Q})
\]

for \( i \leq j - 1 \) (except if \( M \) is of dimension 2 and non-orientable, in which case we require \( i \leq \frac{d}{2} \)).

We actually give a better range that depends on \( M \) as well as \( \lambda \), described by functions \( f_{M, \lambda}(j) : \mathbb{N}_0 \to \mathbb{N}_0 \) defined in Equation 1, where \( \mathbb{N}_0 \) denotes the non-negative integers. The use of rational coefficients is essential in many parts of the argument but not all. See Remark 3.12 for a discussion of what results hold with integral coefficients.

In general, the isomorphism of Theorem 1.3 is given by a transfer map which is described in Definition 5.2. When \( M \) is the interior of a manifold with non-empty boundary one can also define a stabilization map \( t : W_{1; \lambda}(M) \to W_{1; \lambda+1}(M) \) given by “bringing a particle in from infinity,” described in Definition 3.1. The stabilization map induces an isomorphism on rational homology in the same range as the transfer map. Our result uses homological stability for configuration spaces of unordered distinct particles as input. It does not use homological stability for bounded symmetric powers and hence gives a new proof of Theorem 1.6 of \([KM13b]\) with an improved range. Indeed, for orientable manifolds the range one obtains for \( \text{Sym}^\leq_k(M) \) is \( \ast \leq k \) if \( \dim M > 2 \) and \( \ast \leq k - 1 \) if \( \dim M = 2 \).

**1.1. Motivic motivation for Conjecture F.** Conjecture F was inspired by Theorem 1.30 of \([VW12]\) which can be thought of as Conjecture F’s motivic analogue. As an abelian group, the Grothendieck ring of varieties is defined as the quotient of the free abelian group on the set of isomorphism classes of varieties, modulo the relation \([X] = [Z] + [X\setminus Z] \) whenever \( Z \subset X \) is a closed subvariety. The ring structure is induced by Cartesian product. Let \( L \) denote the affine line and let \( X \) be a variety of dimension \( d \). In Theorem 1.30, Vakil and Wood prove that the sequence \([W_{1; \lambda}(X)]/\mathbb{L}]^\ast \) converges in the Grothendieck ring of varieties localized at \( \mathbb{L} \) and completed with respect to the dimensional filtration.

Conjecture F is part of a larger question concerning the relationship between homological stability and stability in the Grothendieck ring of varieties. For many sequences of varieties with homological stability, Vakil and Wood were able to prove that the corresponding elements in the Grothendieck ring stabilize. For \( W_{1; \lambda}(X) \) the corresponding elements in the Grothendieck ring also stabilize, but homological stability was previously not known. There is a close but not exact relationship between the singular homology of a complex variety and its corresponding element in the Grothendieck ring. Conjecture F is obtained from the idea that one should expect homological stability in situations where there is stability in the Grothendieck ring and vice versa. Using motivic zeta functions and a heuristic they dub “Occam’s razor for Hodge structures,” Vakil and Wood developed a procedure for predicting rational Betti numbers from elements of the Grothendieck ring. This heuristic was designed to explain the apparent correlation between the two types of stability and give a prediction of the stable homology. Although there are some examples where Vakil and Wood’s predictions of the stable homology is incorrect, see e.g. \([KM13c]\) and \([Tom13]\), we know of no examples where they make incorrect predictions regarding whether a sequence of spaces has rational homological stability. It would be interesting to know under what conditions stability in the Grothendieck ring is in fact equivalent to rational homological stability.

What about the stable homology? As noted before, Vakil and Wood’s algebro-geometric approach of motivic zeta functions and “Occam’s razor for Hodge structures” does not always correctly predict the limiting rational Betti numbers. The homotopy theoretic technique of “scanning” (see \([Seg73]\)
and [McD75]) also fails to provide an answer in general, though for bounded symmetric powers it can still be made to work (see [Kal01] and [KM13b]). The problem is that in general the spaces $W_{1,\lambda}(M)$ cannot be characterized by local conditions. We would be very interested in any new techniques that shed light into the stable homology groups of the spaces $W_{1,\lambda}(M)$.

1.2. Outline. In Section 2, we describe a spectral sequence for computing compactly supported cohomology associated to an open filtration. In Section 3, we prove Theorem 1.3 when $M$ is an even dimensional orientable manifold which is the interior of a manifold with non-empty boundary. In odd dimensions or when $M$ is not orientable, there are extra complications stemming from the fact that $W_{1,j+1,\lambda}(M)$ is not orientable. In Section 4 we describe how to modify the proof to address these orientation issues. In Section 5, we discuss how to remove the hypothesis that $M$ is the interior of a manifold with non-empty boundary.

1.3. History of this paper. The first two authors and the third author independently proved Conjecture F in [KM13a] and [Tra13] respectively. This paper was obtained by merging those two preprints. Using ideas from both proofs allowed us to streamline the exposition and slightly improve the homological stability ranges.

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2. Compactly supported cohomology

The use of compactly supported cohomology for proving homological stability results was pioneered by Arnol’d in [Arn70]. In this section we review basic properties of compactly supported cohomology and describe a spectral sequence associated to an open filtration. If $N_j$ is a sequence of orientable manifolds each of dimension $n_j$, then $H_*(N_j) \cong H_*(N_{j+1})$ for $* \leq r_j$ if and only if $H_*^c(N_j) \cong H_*^{c+n_j+1-n_j}(N_{j+1})$ for $* \geq n_j - r_j$. Thus, for manifolds homological stability is equivalent to stability with a shift for compactly supported cohomology. It is sometimes more convenient to use compactly supported cohomology for the following reason. Suppose we have a sequence $X_j$ of filtered spaces; then one can often leverage stability for the filtration differences $F_pX_j \setminus F_{p-1}X_j$ to prove stability for the $X_j$. This can be done using the following long exact sequence in compactly supported cohomology (for example see III.7.6 of [Ive86]) and the subsequent spectral sequence derived from it.

**Proposition 2.1.** Let $X$ be a locally compact Hausdorff space and $C \subset X$ a closed subspace with open complement $U = X \setminus C$. There is a long exact sequence in compactly supported cohomology

$$\cdots \to H_*^c(U) \to H_*^c(X) \to H_*^c(C) \to H_*^{c+1}(U) \to \cdots$$

The same holds for compactly supported cohomology with coefficients in a local coefficient system on $X$.

Iterating this gives the following spectral sequence associated to an open filtration. We are unaware of a reference so we sketch a proof.

**Proposition 2.2.** Let $X$ be a locally compact Hausdorff space and

$$\cdots = U_{M-1} = U_M = \emptyset \subset U_{M+1} \subset \cdots \subset X = U_N = U_{N+1} = \cdots$$

be an increasing sequence of open subsets of $X$. Then there is a spectral sequence converging to $H_*^{c+q}(X)$ with $E^1$-page given by

$$E^1_{p,q} = H_*^{c+q}(U_p \setminus U_{p-1})$$

There is a similar spectral sequence for compactly supported cohomology with coefficients in any local coefficient system on $X$. It is natural with respect to open embeddings compatible with the filtrations.
Proof. The idea is to splice together for each $i$ the long exact sequences for the inclusions of a closed set and its complement $U_i \setminus U_{i-1} \hookrightarrow U_i \hookleftarrow U_{i-1}$, given by:

$$\cdots \to H_c^*(U_{i-1}) \to H_c^*(U_i) \to H_c^*(U_i \setminus U_{i-1}) \to \cdots$$

We can invoke Proposition 2.1 in this situation because open subsets of a locally compact Hausdorff space are again locally compact Hausdorff spaces. Next consider the following exact couple

$$\begin{array}{ccc}
A & \xleftarrow{i} & A \\
\downarrow{k} & & \downarrow{j} \\
E & & A
\end{array}$$

with $(a, b)$ denoting the shift in bigrading. Here $i$ is the sum of the maps $H_c^*(U_{p-1}) \to H_c^*(U_p)$ induced by the inclusion of open subsets, $j$ is induced by the restriction map $H_c^*(U_p) \to H_c^*(U_p \setminus U_{p-1})$ along the closed inclusion $U_p \setminus U_{p-1} \hookrightarrow U_p$ and $k$ is given by the boundary maps $H_c^*(U_p \setminus U_{p-1}) \to H_c^*(U_{p-1})$.

The general machinery of exact couples gives us a spectral sequence with

$$E^1_{p,q} = H^p_c(U_p \setminus U_{p-1})$$

which will have homological differentials, i.e. $d_r$ is of bidegree $(-r, r-1)$.

To check that this converges and compute what it converges to, it suffices to note that a spectral sequence of an exact couple has $E^\infty$-page equal to the associated graded of a filtration $F^\ast A_\infty, \ast$ of $A_\infty, \ast$ if there are only finite many $p$ such that $i : A_{p,q} \to A_{p+1,q-1}$ is not an isomorphism and $A_{\infty, \ast} = 0$. In our case $A_\infty, \ast = H_c^*(X)$ and the condition for convergence holds, because $A_{p,q}$ is 0 for $p$ sufficiently small and $A_{p,q} = H^p_c(X)$ for $p$ sufficiently large. \hfill $\Box$

3. THE PROOF FOR OPEN ORIENTED MANIFOLDS OF EVEN DIMENSION

In this section we prove homological stability for $W_{1,1}\lambda(M)$ where $M$ is a connected oriented manifold of even dimension $d = 2n \geq 2$ that is the interior of a manifold with non-empty boundary.

We start by defining the stabilization map. Let $M$ be the interior of $\tilde{M}$, a connected manifold with non-empty boundary $\partial M$. We do not require $\tilde{M}$ to be compact. Let $D_k$ denote the open $k$-disk and $\bar{D}_k$ denote the closed $k$-disk. Pick an embedding $\phi : \bar{D}^{d-1} \hookrightarrow \partial \tilde{M}$ and a homeomorphism

$$\psi : \text{int}(\tilde{M} \cup_\phi \bar{D}^{d-1} \times [0, 1)) \to M$$

whose inverse is isotopic to the inclusion of $M$ into $\text{int}(\tilde{M} \cup_\phi \bar{D}^{d-1} \times [0, 1))$.

**Definition 3.1.** The stabilization map $t : \mathbb{R}^d \times W_{1,1}\lambda(M) \to W_{1,1}\lambda(M)$ is defined as follows: For $z \in \mathbb{R}^d$ and $\xi \in W_{1,1}\lambda(M)$, let $\xi \cup z$ be the element of $W_{1,1}\lambda(\text{int}(\tilde{M} \cup_\phi \bar{D}^{d-1} \times [0, 1]))$ given by $\xi$ in $M$ and $z$ in $\mathbb{R}^d \cong \bar{D}^{d-1} \times (0, 1)$. Define $t$ by the formula $t(z, \xi) = \hat{\psi}(\xi \cup z)$ where

$$\hat{\psi} : W_{1,1+1}\lambda(\text{int}(\tilde{M} \cup_\phi \bar{D}^{d-1} \times [0, 1])) \to W_{1,1}\lambda(M)$$

is the map induced by applying $\psi$ to every point in the configuration.

Note that this map depends on a choice of embedding and homeomorphism. However, up to homotopy, it only depends on choice of component of $\partial \tilde{M}$ (and if that component of $\partial \tilde{M}$ is orientable, a choice of orientation of $\partial \tilde{M}$). Since $\mathbb{R}^d$ is contractible, $H_c(\mathbb{R}^d \times W_{1,1}\lambda(M)) \cong H_c(W_{1,1}\lambda(M))$. However, our proof of homological stability will use compactly supported cohomology. From this perspective, the copy of $\mathbb{R}^d$ is relevant. In particular, it makes the stabilization map an open embedding so it induces a map on compactly supported cohomology. In a similar fashion, one can define stabilization maps for the strata $S_{\lambda}(M)$. To state the main result of this section, we need the following definition.

**Definition 3.2.** A manifold $M$ is said to satisfy condition $(*)_a$ for $a < \dim M - 1$ if $\hat{H}_c(M; \mathbb{Q}) = 0$ for $i \leq a$.

The goal of this section is to prove the following proposition.
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**Proposition 3.3.** Let $M$ be a connected oriented manifold of even dimension $d = 2n \geq 2$ that is the interior of a manifold with non-empty boundary. The stabilization map

$$ t_* : H_i(\mathbb{R}^d \times W_{1, \lambda}(M); \mathbb{Q}) \to H_i(W_{1, \lambda}(M); \mathbb{Q}) $$

is an isomorphism for $i < f_{M, \lambda}(j)$ and a surjection for $i = f_{M, \lambda}(j)$, where $f_{M, \lambda}(j)$ is defined as follows.

(1) \[ f_{M, \lambda}(j) = \begin{cases} 
  j + k & \text{if } \dim M = d > 2; \\
  j + k - 1 & \text{if } \dim M = 2 \text{ and } M \text{ is orientable}; \\
  (j + k)/2 & \text{if } \dim M = 2 \text{ and } M \text{ is not orientable}; \\
  (a + 1)(j + k) - 1 & \text{if condition } (\ast)_a \text{ holds for } a \geq 1 \text{ and } M \text{ is orientable}.
\end{cases} \]

We will not address the case that $M$ is not orientable in this section and so the range of $(j + k)/2$ will not be discussed until the following section. In Lemma 5.2, we will show that the stabilization map is always injective in homology which will imply that it is an isomorphism for $i = f_{M, \lambda}(j)$.

Our approach for proving Proposition 3.3 will be to filter $W_{1, \lambda}(M)$. The filtration differences in this filtration will consist of disjoint unions of spaces of the form $S_{V}(M)$. We can assemble this data into a spectral sequence for compactly supported cohomology using Proposition 2.2. Using homological stability for the filtration differences and rational Poincaré duality we will see that the $W_{1, \lambda}(M)$ satisfy stability in compactly supported cohomology. Again by rational Poincaré duality this will be equivalent to showing that $W_{1, \lambda}(M)$ satisfies rational homological stability.

Using the homological stability results of Church [Chu12] and Randal-Williams [RW13], we prove that the stabilization map induces isomorphisms in homology on the components of each of the filtration differences. This is sometimes referred to as homological stability for colored configuration spaces.

**Lemma 3.4.** Let $\lambda$ be a partition with $i$ 1’s. If $M$ is orientable, the map

$$ t_* : H_i(\mathbb{R}^d \times S_{\lambda}(M); \mathbb{Q}) \to H_i(S_{1, \lambda}(M); \mathbb{Q}) $$

is an isomorphism in the following ranges: (i) $\ast \leq i$ if $M$ is of dimension $d > 2$, (ii) $\ast < i$ if $M$ is of dimension $d = 2$, and (iii) $\ast < (a + 1)i$ if condition $(\ast)_a$ holds.

**Proof.** Let $\lambda'$ be the partition such that $1^{\lambda'} = \lambda$ and let $r$ be the cardinality of $\lambda'$, i.e. $\lambda'$ is given by $m_1 + \cdots + m_r$ with $m_i \geq 2$ for all $1 \leq i \leq r$. There is a fiber bundle:

$$ S_1(M \{r \text{ points}\}) \to S_{\lambda}(M) \to S_{\lambda'}(M). $$

The map $S_{\lambda}(M) \to S_{\lambda'}(M)$ is the map that forgets all of the points labeled by the number 1. The stabilization map induces a map from the Serre spectral sequence for this fibration (multiplied with $\mathbb{R}^d$) to the Serre spectral sequence for the version of this fibration associated to $S_{1, \lambda}(M)$.

The result now follows by spectral sequence comparison and the homological stability ranges of Church and Randal-Williams: a range $\ast < i$ for all dimensions $\geq 2$ from Corollary 3 of [Chu12], a range $\ast < i$ for all dimensions $\geq 3$ from Theorem B of [RW13] and the improved range with vanishing reduced Betti numbers from Proposition 4.1 of [Chu12].

Two remarks are in order. Firstly, Church’s results concern the transfer map, not the stabilization map. This is not an issue as the proof of Lemma 5.2 shows that these maps are rationally mutually inverse in the stable range (also see Section 7 of [RW13] and Lemma 2.2 of [Dol62]). Secondly, if $M$ has the property that $H_i(M; \mathbb{Q}) = 0$ for $i \leq a$ with $a < \dim M - 1$, then by Mayer-Vietoris $M \{r \text{ points}\}$ has the same property.

The spaces $W_{\lambda}(M)$ have a filtration whose filtration differences consist of disjoint unions of spaces of the form $S_{\lambda'}(M)$. We will now describe this filtration. An elementary collapse of a partition $\lambda$ is a partition $\lambda'$ which is identical to $\lambda$ except that two integers have been replaced by their sum. A partition $\lambda'$ is a collapse of $\lambda$ if $\lambda'$ can be constructed from $\lambda$ by a sequence of elementary collapses. In this case we write $\lambda' \leq \lambda$. For example $1 + 2 + 2 + 4 \leq 1 + 1 + 1 + 2 + 4$. For $\lambda$ a partition of $\lambda$, let $\text{col}_p(\lambda)$ be the set of collapses of $1^k$ by $p$ elementary collapses that are not collapses of $\lambda$. In other words, $\text{col}_p(\lambda)$ is the set of partitions $\lambda' \leq \lambda$ with cardinality $k - p$.️
Example 3.5. If \( \lambda = 1 + 3 \), then \( \text{col}_0(\lambda) = \{1 + 1 + 1 + 1\} \), \( \text{col}_1(\lambda) = \{1 + 1 + 2\} \), \( \text{col}_2(\lambda) = \{2 + 2\} \), and \( \text{col}_3(\lambda) = \{\emptyset\} \) for \( p \geq 3 \). In particular, \( 1 + 3 \not\in \text{col}_2(\lambda) \).

The filtration of \( W_\lambda(M) \) that we will describe will have filtration differences whose components are indexed by the sets \( \text{col}_p \).

Definition 3.6. For a partition \( \lambda \), manifold \( M \) and \( p \geq 0 \), let

\[
\mathcal{S}_\lambda[p] = \bigcup_{\lambda' \in \text{col}_p(\lambda)} S_{\lambda'}(M) \quad \text{and} \quad U_p = \bigcup_{q \leq p} \mathcal{S}_\lambda[q].
\]

Note that \( U_p \) is also equal to

\[
W_\lambda(M) \setminus \left( \bigcup_{\lambda' \in \text{col}_p(\lambda) \text{ for } l \geq p+1} W_{\lambda'}(M) \right)
\]

so in particular the \( U_p \) are open. This is an increasing filtration with \( U_p = W_\lambda(M) \) for \( p \geq k \), \( U_0 = S_1(M) \) and \( U_p \) empty for \( p < 0 \). Crossing with \( \mathbb{R}^d \) gives a filtration of \( \mathbb{R}^d \times W_\lambda(M) \). Using Proposition 2.2 applied to the filtrations of \( W_\lambda(M) \) and \( \mathbb{R}^d \times W_\lambda(M) \) gives the following spectral sequences.

Lemma 3.7. There exists a first quadrant spectral sequence converging to \( H^{p+q}_c(W_\lambda(M)) \) with \( E^1 \)-page

\[
E^1_{p,q} = H^{p+q}_c(\mathcal{S}_\lambda[p]).
\]

Similarly, there is a first quadrant spectral sequence converging to \( H^{p+1}_c(\mathbb{R}^d \times W_\lambda(M)) \) with \( E^1 \)-page

\[
E^1_{p,q} = H^{p+q}_c(\mathbb{R}^d \times \mathcal{S}_\lambda[p]).
\]

Lemma 3.8. The stabilization map \( t_* : H^*_c(\mathbb{R}^d \times W_\lambda(M)) \to H^*_c(W_1 \lambda(M)) \) respects the spectral sequences of Lemma 3.7. Moreover, the maps on the \( E^1 \)-page are induced by the stabilization maps

\[
t_* : H^*_c(\mathbb{R}^d \times \mathcal{S}_\lambda(M)) \to H^*_c(\mathcal{S}_1 \lambda(M)).
\]

Proof. Recall that open embeddings induce maps on compactly supported cohomology via extension by zero. The stabilization map is an open embedding compatible with the filtrations used in Lemma 3.7.

Lemma 3.9. Let \( \lambda \) be a partition of \( k \). The stabilization map induces a bijection between the sets of path components of \( S_{1 \lambda}[p] \) and \( S_{1 \lambda + 1}[p] \) for \( p \leq \frac{k+1}{2} \).

Proof. The map on path components induced by the stabilization map is clearly injective for all \( i \). It is not surjective only when \( S_{1 \lambda + 1}[p] \) has a component of the form \( S_{\lambda'}(M) \) where \( \lambda'' \) is a partition without any 1’s. On the other hand if \( S_{\lambda'}(M) \subset S_{1 \lambda + 1}[p] \) then \( \lambda'' \) consists of at least \( j + k + 1 - 2p \) 1’s. Therefore, the map of components is surjective when \( j + k + 2p \geq 0 \) or equivalently \( p \leq \frac{k+1}{2} \).

It is helpful to combine Lemma 3.4 and Lemma 3.9 into a single statement.

Lemma 3.10. Let \( \lambda \) be a partition of \( k \). The map \( t_* : H_*(\mathbb{R}^d \times S_{1 \lambda}[p]; \mathbb{Q}) \to H_*(S_{1 \lambda + 1}[p]; \mathbb{Q}) \) is an isomorphism in the following ranges: \( * \leq k + j - 2p \) if \( M \) is of dimension \( d = 2 \), \( * < k + j - 2p \) if \( M \) is of dimension \( d = 2 \), and \( * < (a + 1)(k + j - 2p) \). Note that this includes negative \( * \).

The following lemma follows from the existence of relative spectral sequences.

Lemma 3.11. If \( f : E^1_{p,q} \to \mathcal{E}^1_{p,q} \) is a surjection for \( p + q \geq * - 1 \) and an isomorphism for \( p + q \geq * \) then \( f : \mathcal{E}^\infty_{p,q} \to \mathcal{E}^\infty_{p,q} \) is a surjection for \( p + q \geq * - 1 \) and an isomorphism for \( p + q \geq * \).

We now prove Proposition 3.3.
Proof. Note that the spaces $W_{1,λ}(M)$ satisfy rational Poincaré duality because they are quotients of orientable manifolds by finite groups acting via orientation preserving maps (this uses that the dimension of $M$ is even). In other words, the spaces $W_{1,λ}(M)$ are the underlying space of orientable orbifolds. Using rational Poincaré duality, we see that it suffices to show that the stabilization map

$$t_* : H_c^*(\mathbb{R}^d \times W_{1,λ}(M); \mathbb{Q}) \to H_c^*(W_{1,λ+1}(M); \mathbb{Q})$$

is an isomorphism for $* > \dim(W_{1,λ+1}(M)) - f_{M,λ}(j)$ and a surjection for $* = \dim(W_{1,λ+1}(M)) - f_{M,λ}(j)$.

By Lemma 3.7, there are spectral sequences

$$E^1_{p,q} = H_c^{p+q}(\mathbb{R}^d \times S_{1,λ}[p]; \mathbb{Q}) \Rightarrow H_c^{p+q}(\mathbb{R}^d \times W_{1,λ}(M); \mathbb{Q})$$

and

$$E^1_{p,q} = H_c^{p+q}(S_{1,λ+1}[p]; \mathbb{Q}) \Rightarrow H_c^{p+q}(W_{1,λ+1}(M); \mathbb{Q}).$$

By Lemma 3.8, there is a map of spectral sequence $t_* : E^1_{p,q} \to E^1_{p,q}$ which on $E^\infty$ is the stabilization map

$$t_* : H_c^*(\mathbb{R}^d \times W_{1,λ}(M); \mathbb{Q}) \to H_c^*(W_{1,λ+1}(M); \mathbb{Q}).$$

First consider the case of manifolds $M$ of dimension $d = 2n > 2$. Take $p$ and $q$ with $p + q \geq \dim(W_{1,λ+1}(M)) - f_{M,λ}(j) = d(j + k + 1) - (j + k)$. We are interested in proving that the stabilization map

$$t_* : H_c^{p+q}(\mathbb{R}^d \times S_{1,λ}[p]; \mathbb{Q}) \to H_c^{p+q}(S_{1,λ+1}[p]; \mathbb{Q})$$

is an isomorphism. Using Poincaré duality this is equivalent to the map

$$H_{d(j+k+1-p)-p-q}(\mathbb{R}^d \times S_{1,λ}[p]; \mathbb{Q}) \to H_{d(j+k+1-p)-p-q}(S_{1,λ+1}[p]; \mathbb{Q})$$

being an isomorphism. From Lemma 3.10 we know this is the case if $d(j+k+1-p) - p - q \leq j + k - 2p$, i.e. if $(d-1)p + q \geq d(j + k + 1) - (j + k)$ since $d - 1 \geq 1$. We conclude that the map $E^1_{p,q} \to E^1_{p,q}$ is an isomorphism for $p + q \geq d(j + k + 1) - (j + k)$.

By Lemma 3.11, the stabilization map

$$t_* : H_c^*(\mathbb{R}^d \times W_{1,λ}(M); \mathbb{Q}) \to H_c^*(W_{1,λ+1}(M); \mathbb{Q})$$

is an isomorphism for $* > \dim(W_{1,λ+1}(M)) - (j + k)$ and a surjection for $* = \dim(W_{1,λ+1}(M)) - (j + k)$.

By using the homological stability ranges for manifolds of dimension 2 or manifolds satisfying condition $(*)_a$, we obtain the other stability ranges. 

\[\Box\]

Remark 3.12. If $dM > 2$, we are forced to work with rational coefficients because the spaces $W_{1,λ}(M)$ do not have integral Poincaré duality. However, if $dM = 2$ we can prove stability integrally, because $W_{1,λ}(M)$ is a manifold and hence has integral Poincaré duality. Almost all of the arguments of this section go through unchanged to prove that $H_*(W_{1,λ}(M); \mathbb{Z})$ stabilizes if $dM = 2$. The one modification needed is the following. Instead of using the results of Church in [Chu12] and Randal-Williams in [RW13] on rational homological stability for $S_1(M)$, we use the integral results of Segal; in Proposition A.1 of [Seg79] he proved that $t : S_1(M) \to S_{1,λ}$ induces an isomorphism on integral homology for $* \leq j/2$. Therefore, we have that $t_* : H_*(W_{1,λ}(M); \mathbb{Z}) \to H_*(W_{1,λ+1}(M); \mathbb{Z})$ is an isomorphism for $* \leq (j + k)/2$ if $dM = 2$.

When we consider closed manifolds in Section 5, the use of rational coefficients will also be unavoidable. In fact, from the presentation of the spherical braid group given in [FVB62], one sees that $H_1(W_{1,λ}(\mathbb{C}P^1); \mathbb{Z}) = \mathbb{Z}/(2j + 2)\mathbb{Z}$. Hence integral homological stability fails for closed manifolds, even in dimension two.

We also note that these techniques show that the spaces $S_1(M)$ and $W_{1,λ}(M)$ have stability for appropriately shifted integral compactly supported cohomology provided $M$ is a connected manifold that is the interior of a manifold with non-empty boundary and has dimension at least 2.
4. The proof for general open manifolds

The goal of this section is to generalize Proposition 3.3 to the case of open manifolds that are not necessarily orientable or even dimensional. If $M$ is not orientable or not even dimensional then $W_\lambda(M)$ is not orientable. In the previous section, we regularly used Poincaré duality. These proofs still work if one appropriately modifies the statements of the theorems using local coefficient systems.

Since $W_\lambda(M)$ is the underlying space of an orbifold, there is a rational orientation local coefficient system $\mathcal{O}$ such that $H_* (W_\lambda(M); \mathbb{Q}) \cong H_*^{\dim (W_\lambda(M)) - s} (W_\lambda(M); \mathcal{O})$. To compute $H^*_c(W_\lambda(M); \mathcal{O})$, we use the filtration from Definition 3.6. On the spaces in the filtration and on the filtration differences, we use local coefficients which are pullbacks of $\mathcal{O}$ under the inclusions into $W_\lambda(M)$. Lemma 3.7 generalizes as follows.

**Lemma 4.1.** Let $\iota : S_\lambda[p] \to W_\lambda(M)$ be the inclusion map. There exists a first quadrant spectral sequence converging to $H^{p+q}_c(W_\lambda(M); \mathcal{O})$ with $E_1$-page $E_{p,q}^1 = H^{p+q}_c(S_\lambda[p]; \iota^* \mathcal{O})$. A similar spectral sequence also exists computing $H^{p+q}_c(\mathbb{R}^d \times W_\lambda(M); \mathcal{O'})$ with $\mathcal{O}'$ the orientation local coefficient system on $\mathbb{R}^d \times W_\lambda(M)$.

Using these spectral sequences, we see that to prove homological stability for symmetric complements, it suffices to show that $t_* : H_*(\mathbb{R}^d \times S_\lambda(M); \mathcal{O}' \otimes \iota^* \mathcal{O}') \to H_*(S_\lambda(M); \mathcal{O}_\lambda \otimes \iota^* \mathcal{O})$ is an isomorphism in a range. The spaces $S_\lambda(M)$ are manifolds and hence have Poincaré duality. Let $\mathcal{O}_\lambda$ denote the orientation local coefficient system of $S_\lambda(M)$ and $\mathcal{O}'_\lambda$ denote the orientation local system on $\mathbb{R}^d \times S_\lambda(M)$. By Poincaré duality for the strata, we see that stability for the groups $H_*(S_\lambda(M); \mathcal{O}_\lambda \otimes \iota^* \mathcal{O})$ is the relevant generalization of Lemma 3.4.

**Lemma 4.2.** Let $\lambda$ be a partition with $i$ 1’s and let $M$ be a connected $d$ dimensional manifold which is the interior of a manifold with non-empty boundary. The map

$$t_* : H_*(\mathbb{R}^d \times S_\lambda(M); \mathcal{O}'_\lambda \otimes \iota^* \mathcal{O}') \to H_*(S_\lambda(M); \mathcal{O}_\lambda \otimes \iota^* \mathcal{O})$$

is an isomorphism in the following ranges: (i) $* \leq i$ if $M$ is of dimension $d > 2$, (ii) $* < i$ if $M$ is of dimension $d = 2$ and $M$ is orientable, (iii) $* \leq i/2$ if $M$ is of dimension $d = 2$ and $M$ is not orientable, and (iv) $* < (\alpha + 1)i$ if condition $(\ast)_a$ holds and $M$ is orientable.

**Proof.** Consider the fiber bundle

$$S_\lambda(M\setminus\{r \text{ points}\}) \to S_\lambda(M) \to S_\lambda(M)$$

used in the proof of Lemma 3.4. There is a twisted Serre spectral sequence for this fibration converging to the homology of $H_*(S_\lambda(M); \mathcal{O}_\lambda \otimes \iota^* \mathcal{O})$. This has $E_2$-page given by the homology of $S_\lambda(M)$ with coefficients in a graded local system $\mathcal{F}_\lambda$. This graded local system $\mathcal{F}_\lambda$ is a bundle of graded $\mathcal{Q}$ vector spaces with fibers isomorphic to $H_*(S_\lambda(M\setminus\{r \text{ points}\})(M); f^*(\mathcal{O}_\lambda \otimes \iota^* \mathcal{O}))$. Here $f$ is the inclusion of a fiber into the total space.

We now explain why the local system $f^*(\mathcal{O}_\lambda \otimes \iota^* \mathcal{O})$ is in fact trivial. Consider $\gamma \in \pi_1(S_\lambda(M))$. There are two natural maps $d_1, d_2 : \pi_1(S_\lambda(M)) \to H_1(M; \mathbb{Z})$ given by viewing $\gamma$ as a collection of loops in $M$. The first map $d_1$ remembers the multiplicities of the particles while the second one does not. Let $\mathcal{O}_M : H_1(M; \mathbb{Z}) \to \mathbb{Z}_2 = \{1, -1\}$ be the monodromy associated to the orientation local system on $M$. Suppose $\lambda$ has $n_1$ 1’s, $n_2$ 2’s, etc. Let $p_m : \pi_1(S_\lambda(M)) \to \mathcal{G}_{n_m}$ be the map that remembers the permutations associated to the paths of particles of multiplicity $m$. Let $s_2 : \pi_1(S_\lambda(M)) \to \mathbb{Z}_2$ be given by $s_2(\gamma) = \prod \epsilon(p_m(\gamma))$ with $\epsilon$ the sign homomorphism. Likewise define $s_1$ by the formula $s_1(\gamma) = \prod \epsilon(p_m(\gamma))^m$. The local system $\iota^* \mathcal{O}$ can be described as the local system with monodromy around the loop gamma given by $\mathcal{O}_M(d_1(\gamma)) s_1(\gamma)^d$. The local system $\mathcal{O}_\lambda$ on the other hand associates to $\gamma$ the number $\mathcal{O}_M(d_2(\gamma)) s_2(\gamma)^d$. These two local systems agree on loops where only particles with odd multiplicities move. Therefore, they agree on the image of the fundamental group of a fiber and so $f^*(\mathcal{O}_\lambda \otimes \iota^* \mathcal{O})$ is the trivial local coefficient system $\mathcal{Q}$.

The rest of the proof follows the pattern of Lemma 3.4. The stabilization map induces a map from the Serre spectral sequence for this fibration cross $\mathbb{R}^d$ to the Serre spectral sequence for fibration associated to $S_1 \lambda(M)$. Since the local systems on the fibers are trivial, homological stability for the spaces
$\tau_1(M)$ and spectral sequence comparison complete the proof. We note that Church’s homological stability result only applies to orientable manifolds and Randal-Williams’ range of $* \leq i$ only applies in dimensions $> 2$. For non-orientable surfaces we instead invoke the homological stability range given by Segal in Proposition A.1 of [Seg79].

The rest of the arguments in Section 3 apply with little modification to give the following proposition.

**Proposition 4.3.** Let $M$ be a connected manifold $M$ of dimension at least 2 that is the interior of a manifold with boundary. The stabilization map $t_\ast: H_\ast(Q \times W_{1,\lambda}(M); \mathbb{Q}) \to H_\ast(W_{1,\lambda}(M); \mathbb{Q})$ is an isomorphism for $i < f_{M,\lambda}(j)$ and a surjection for $i = f_{M,\lambda}(j)$, the function given in Equation 1.

5. The proof for closed manifolds by puncturing

In this section, we prove homological stability for the spaces $W_{1,\lambda}(M)$ for $M$ closed. One cannot define stabilization maps for closed manifolds as there is no way to add an extra point. Instead we use the so-called transfer map. Our proof is similar to that used by Randal-Williams in Section 9 of [RW13] to leverage homological stability for configuration spaces of particles in open manifolds to prove homological stability for configuration spaces of particles in closed manifolds. We will first recall the definition of the transfer map $\tau: H_\ast(W_{1,\lambda}(M); \mathbb{Q}) \to H_\ast(W_{1,\lambda}(M); \mathbb{Q})$. We then prove that the transfer map induces an isomorphism in the same range as the stabilization map when the manifold is open. Using an augmented semisimplicial space, we describe a spectral sequence computing $H_\ast(W_{1,\lambda}(M); \mathbb{Q})$ in terms of $H_\ast(W_{1,\lambda}(N); \mathbb{Q})$ with $N$ being $M$ minus a finite number of points. The transfer map will respect this spectral sequence. The theorem will now follow by comparing the spectral sequence for $W_{1,\lambda}(M)$ with the one for $W_{1,\lambda+1}(M)$.

Let $\lambda$ be a partition of $k$. Let $\tilde{S}_\lambda(M)$ and $\tilde{S}_\lambda(M)$ be ordered versions of the symmetric complement and stratum. That is, these spaces are defined as the inverse image of the projection $M^k \to \text{Sym}_k(M)$ of the spaces $W_\lambda(M)$ and $S_\lambda(M)$. For $i \leq j$, let $\text{del}_{i,j} : \tilde{W}_{1,\lambda}(M) \to \tilde{W}_{1,\lambda}(M)$ be the map which deletes the last $j - i$ points of $\tilde{W}_{1,\lambda}(M)$. This makes sense since the points of $\tilde{W}_{1,\lambda}(M)$ are ordered and $1' \lambda' \leq 1\lambda$ if $\lambda' \leq \lambda$. The map $\text{del}_{i,j}$ is $\mathfrak{S}_{i+k}$-equivariant. Here $\mathfrak{S}_{i+k}$ acts on $\tilde{W}_{1,\lambda}(M)$ via the inclusion of $\mathfrak{S}_{i+k}$ into $\mathfrak{S}_j$ induced by the standard inclusion $\mathfrak{S}_i$ into $\mathfrak{S}_j$. Thus it induces a map $(\text{del}_{i,j})_\ast : H_\ast(W_{1,\lambda}(M); \mathbb{Q}) \mathfrak{S}_{i+k} \to H_\ast(W_{1,\lambda}(M); \mathbb{Q}) \mathfrak{S}_{i+k}$. Here $V_G$ denotes the coinvariants of the action of a group $G$ on a $G$-module $V$. Recall that if $X$ is a $G$-space, $H_\ast(X/G; \mathbb{Q}) \cong H_\ast(X; \mathbb{Q})_G$. Since we are viewing $\mathfrak{S}_{i+k}$ as a subgroup of $\mathfrak{S}_{j+k}$, we get an inclusion $\iota : H_\ast(W_{1,\lambda}(M); \mathbb{Q}) \mathfrak{S}_{j+k} \to H_\ast(W_{1,\lambda}(M); \mathbb{Q}) \mathfrak{S}_{j+k}$. Using this we can define the transfer map.

**Definition 5.1.** The transfer map $\tau_{i,j} : H_\ast(W_{1,\lambda}(M); \mathbb{Q}) \to H_\ast(W_{1,\lambda}(M); \mathbb{Q})$ is defined as

$$(\text{del}_{i,j})_\ast \circ \iota : H_\ast(W_{1,\lambda}(M); \mathbb{Q}) \mathfrak{S}_{j+k} \to H_\ast(W_{1,\lambda}(M); \mathbb{Q}) \mathfrak{S}_{j+k}$$

postcomposed and precomposed with the natural isomorphisms $H_\ast(W_{1,\lambda}(M); \mathbb{Q}) \cong H_\ast(W_{1,\lambda}(M); \mathbb{Q}) \mathfrak{S}_{j+k}$ and $H_\ast(W_{1,\lambda}(M); \mathbb{Q}) \cong H_\ast(W_{1,\lambda}(M); \mathbb{Q}) \mathfrak{S}_{j+k}$ respectively.

For $i = j - 1$, we denote $\tau_{i,j}$ by $\tau$. Since we will not use compactly supported cohomology in this section, we drop the factor of $\mathbb{R}^d$ from the domains of stabilization maps. We now show that $\tau$ induces a rational homology equivalence in a range for $M$ open by proving it is the inverse to the stabilization map in the stable range.

**Lemma 5.2.** Let $M$ be the interior of a connected manifold with non-empty boundary. The stabilization map induces injections $t_\ast : H_\ast(W_{1,\lambda}(M); \mathbb{Q}) \to H_\ast(W_{1,\lambda+1}(M); \mathbb{Q})$ and hence is an isomorphism for $* = f_{M,\lambda}(j)$, the function given in Equation 1. Additionally, the transfer map $\tau : H_\ast(W_{1,\lambda+1}(M); \mathbb{Q}) \to H_\ast(W_{1,\lambda}(M); \mathbb{Q})$ is an inverse to the stabilization map in the range $* \leq f_{M,\lambda}(j)$. In particular, the transfer map is an isomorphism in this range.

**Proof.** Suppose that $\lambda$ is a partition of $k$, then for $1 \leq j \leq 2k$ we let $W_{1,\lambda}(M)$ be the inverse image of $W_\lambda(M)$ in $\text{Sym}_k(M)$ under $t^{-j}$. Fix $i \geq 0$ and set $B_j = H_\ast(W_{1,\lambda-k}(M); \mathbb{Q})$. We then define $\sigma_j : B_{j-1} \to B_j$ to be the map on homology induced by the stabilizing map for $j \geq 1$ and $\sigma_0$ to be
$0 \to B_0$. The transfer gives maps $\tau_{q,p} : B_p \to B_q$. These satisfy $\tau_{q,p} \circ \sigma_p = \tau_{q,p-1} + \sigma_{p-1} \circ \tau_{q-1,p-1}$ and $\tau_{p,p} = \text{id}$. By Lemma 2.2 of [Dol62], the map

$$\bigoplus_{q \leq p} \tau_{q,p} : B_p \to \bigoplus_{0 \leq q \leq p} B_q/\text{im}(\sigma_q)$$

is an isomorphism. Using this one can conclude that $\tau_{p-1,p} \circ \sigma_p$ respects this decomposition and on each summand is multiplication by a non-zero scalar. More precisely, first note that $\tau_{m,m+1} \circ \ldots \circ \tau_{p-1,p} = (p-m)! \tau_{m,p}$ and thus $(p-m)! \tau_{q,p} = \tau_{q,m} \circ \tau_{m,p}$ rationally. We now compute the composition of $\tau_{p-1,p} \circ \sigma_p$ with one of the maps in the decomposition. Let $\pi_q$ be the projection $B_q \to B_q/\text{im}(\sigma_q)$. We have that the projection to the $q$-summand of $B_{p-1}$ is given by the map $\pi_q \circ \tau_{q,p}$. Now note that

$$\pi_q \circ \tau_{q,p-1} \circ \tau_{p-1,p} \circ \sigma_p = (p-q) \pi_q \circ \tau_{q,p} \circ \sigma_p$$

$$= (p-q) \pi_q \circ (\tau_{q,p-1} + \sigma_q \circ \tau_{q-1,p-1})$$

$$= (p-q) \pi_q \circ \tau_{q,p-1}$$

But $\pi_q \circ \tau_{q,p-1}$ is exactly the projection onto the $q$-summand of $B_{p-1}$. This shows that $\tau_{p-1,p} \circ \sigma_p$ is injective and so the stabilization maps induce injections in homology. Since we are working rationally, this also implies that $\tau_{p-1,p}$ is an isomorphism when $\sigma_p$ is. Specializing to $p = k + j$ gives the desired result.  

Recall that a semisimplicial object is defined in the same way as a simplicial object without the data of degeneracy maps. We now describe a semisimplicial space $\tilde{W}_\bullet(\lambda)$ with augmentation to $\tilde{W}_{1/\lambda}(M)$.

**Definition 5.3.** The space of $p$ simplices of $\tilde{W}_\bullet(\lambda)$ is given by

$$\tilde{W}_p(\lambda) = \bigcup_{\{m_0, \ldots, m_p\} \in \delta_{p+1}(M)} \tilde{W}_\lambda(M \setminus \{m_0, \ldots, m_p\})$$

The $i$th face map is induced by the inclusion $M \setminus \{m_0, \ldots, m_p\} \to M \setminus \{m_0, \ldots, m_{p-1}\}$ which fills in the $i$th puncture.

The above construction works equally well for $p = -1$, giving it the structure of an augmented semisimplicial space. Note that $\tilde{W}_{-1}(\lambda) = \tilde{W}_\lambda(M)$. We will show that the augmentation map induces a weak equivalence $||\tilde{W}_\bullet(\lambda)|| \to \tilde{W}_\lambda(M)$. For this reason we will call $\tilde{W}_\bullet(\lambda)$ a resolution of $\tilde{W}_\lambda(M)$. This resolution is useful as $M \setminus \{m_0, \ldots, m_p\}$ is an open manifold and so we will be able to apply Lemma 5.2 levelwise to a semisimplicial chain complex constructed from $\tilde{W}_\bullet(\lambda)$.

To prove that the augmentation is a weak equivalence, we will first recall the definitions of microfibrations and flag sets. We are interested in these definitions since every microfibration with weakly contractible fibers is a weak equivalence and there is an easily checked condition for the contractibility of the geometric realization of a flag set.

**Definition 5.4.** A map $f : E \to B$ is called a microfibration if for $m \geq 0$ and each commutative diagram

$$\begin{array}{ccc}
\{0\} \times D^m & \longrightarrow & E \\
\downarrow & & \downarrow \\
[0,1] \times D^m & \longrightarrow & B
\end{array}$$

there exists an $\epsilon \in (0,1]$ and a partial lift $[0,\epsilon] \times D^m \to E$ making the resulting diagram commute.

The following proposition was proven by Weiss in Lemma 2.2 of [Wei05].

**Proposition 5.5.** A microfibration with weakly contractible fibers is a weak equivalence.

We now define flag sets, a type of simplicial set where $p$-simplices are defined by their vertices.
Definition 5.6. A semisimplicial set $X_\bullet$ is said to be a flag set if the $p$-simplices $X_p$ are a subset of $X_0^{p+1}$ and if an ordered $(p + 1)$-tuple $(v_0, \ldots, v_p)$ forms a $p$-simplex if and only if $(v_i, v_j)$ forms a 1-simplex for all $i \neq j$.

Lemma 5.7. Let $X_\bullet$ be a flag set such that for each finite collection $\{v_1, \ldots, v_N\}$ of 0-simplices there exists a 0-simplex $v$ such that $(v, v)$ is a 1-simplex for all $i$. Then $\|X_\bullet\|$ is weakly contractible.

Proof. Let $f : S^i \to \|X_\bullet\|$ be arbitrary. By simplicial approximation, we can homotope $f$ to a map $g$ which is simplicial with respect to some PL-triangulation of $S^i$. Note that the image of $g$ is contained in a finite subsemisimplicial set $X_\bullet$ of $X_\bullet$ spanned by some set of 0-simplices $\{v_1, \ldots, v_N\}$. By hypothesis the join $\|X_\bullet\| \ast \{v\}$ is a subcomplex of $\|X_\bullet\|$ and thus we can extend the map $g$ to a map $g : \text{Cone}(S^i) \to \|X_\bullet\|$ by sending the cone point to $v$. □

We now prove that the augmentation is a weak equivalence.

Proposition 5.8. The augmentation induces a weak equivalence $\|\tilde{W}_\lambda(\lambda)\| \to \tilde{W}_\lambda(M)$.

Proof. We will prove that this map is a microfibration with contractible fibers. To see that it is a microfibration, suppose we have a map $f : D^n \times [0, 1] \to \tilde{W}_\lambda(M)$ and a lift $\tilde{f}$ to $\|\tilde{W}_\lambda(j)\|$ defined on $D^n \times \{0\}$. Given a point $y \in \tilde{W}_\lambda(M)$, the extra data needed to lift $y$ to $\|\tilde{W}_\lambda(j)\|$ is a semisimplicial coordinate $t \in \text{int}(\Delta^n)$ and a configuration $(m_0, \ldots, m_p) \in \tilde{S}_{p+1}(M)$ such that $m_0, \ldots, m_p$ are disjoint from the points of $y'$. Note that if $y'$ is sufficiently close to $y$, the points $m_0, \ldots, m_p$ will also be disjoint from $y$. Therefore, the data used to lift $y$ will also define a lift of $y'$. Since $f$ is continuous, for any $x \in D^n$ using the configuration and simplicial coordinate associated to $f(x)$ we can define a lift of $f$ on $\{0\} \times [0, \epsilon]$ for some $\epsilon > 0$. Furthermore since $\tilde{f}$ is continuous we can extend this lift to $U_{\epsilon} \times [0, \epsilon/2]$ with $U_{\epsilon}$ a neighborhood of $x$ in $D^n$. This extension is similarly defined using the simplicial coordinate and configuration associated to $\tilde{f}(x')$ to lift the point $f(x', s)$ to a point in $\|\tilde{W}_\lambda(j)\|$. By compactness of $D^n$, we can find one choice of $\epsilon$ such that this construction defines a lift on all of $D^n \times [0, \epsilon]$.

Next we will show that the fibers of the augmentation map are contractible. Note that the fiber of the augmentation map over a configuration $y \in \tilde{W}_\lambda(M)$ is homeomorphic to the geometric realization of the following flag set $F_\bullet(y)$. The set of $p$-simplices of $F_\bullet(y)$ is the underlying set of $\tilde{S}_{p+1}(M \setminus y)$ and the face maps are induced by forgetting the $i$th point. It is clear that $F_\bullet(y)$ is a flag set. It satisfies the conditions of Lemma 5.7 since we can always find a point in $M$ not contained in some fixed finite subset. Therefore the fibers are weakly contractible and so the augmentation is a weak equivalence. □

We now prove that the transfer map induces a homology equivalence in a range for arbitrary manifolds of dimension at least 2.

Theorem 5.9. Let $M$ be any connected manifold of dimension at least 2. The transfer map $\tau : H_i(W_{1,\lambda+1}(M); \mathbb{Q}) \to H_i(W_{1,\lambda}(M); \mathbb{Q})$ is an isomorphism for $i \leq f_{M,\lambda}(j)$, the function given in Equation 1.

Proof. First we will extend the transfer map to the symmetric group invariant singular chains of the resolution. Applying rational singular chains and using that geometric realization commutes with singular chains up to quasi-isomorphism gives us a semisimplicial chain complex $C_\bullet(W_\bullet(j); \mathbb{Q})$ such that the augmentation $\|C_\bullet(W_\bullet(j); \mathbb{Q})\| \to C_\bullet(\tilde{W}_\lambda(M); \mathbb{Q})$ is a quasi-isomorphism. Applying $S_{k+j}$-coinvariants levelwise we get a semisimplicial complex with augmentation to a chain complex with homology $H_\bullet(W_{1,\lambda}(M); \mathbb{Q})$ and level $p$ having homology given by

$$\bigoplus_{\{m_0, \ldots, m_p\} \in \tilde{S}_{p+1}(M)} H_\bullet(W_{1,\lambda}(M \setminus \{m_0, \ldots, m_p\}); \mathbb{Q})$$

Applying the construction of the transfer map levelwise to the augmented semisimplicial chain complex gives us a semisimplicial chain map $(\tau_\bullet)_\lambda : C_\bullet(W_\lambda(j + 1); \mathbb{Q}) \to C_\bullet(W_\bullet(j); \mathbb{Q})$ inducing the transfer map on homology levelwise. Recall that there is a spectral sequence converging to the homology of a
geometric realization of a semisimplicial chain complex in terms of the homology of the levels. If $A_\bullet$ is a semisimplicial chain complex, this spectral sequence has $E^1_{pq} = H_q(A_p)$. The transfer map induces a map between the spectral sequence for $1^j\lambda$ and $1^{j+1}\lambda$. Since $M\{m_0,\ldots,m_p\}$ are connected manifolds that are the interior of a manifold with non-empty boundary, the previous lemma implies that $(\tau_{\lambda})_\bullet$ induces an isomorphism on $E^1_{pq}$ for $q \leq f_{M,\lambda}(j)$. By a spectral sequence comparison theorem, the transfer map $\tau: H_*(W_{1^{j+1}\lambda}(M);\mathbb{Q}) \to H_*(W_{1^j\lambda}(M);\mathbb{Q})$ is an isomorphism in the range $* \leq f_{M,\lambda}(j)$. □

References


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