SUMMATION FORMULAE FOR ELLIPTIC HYPERGEOMETRIC SERIES

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ABSTRACT. Several new identities for elliptic hypergeometric series are proved. Remarkably, some of these are elliptic analogues of identities for basic hypergeometric series that are balanced but not very-well-poised.

1. INTRODUCTION

Recently there has been much interest in elliptic hypergeometric series [3–7, 9, 10, 13–17, 19–25]. The simplest examples of such series are of the type

$$
\begin{aligned}
(1.1)\quad {}_{r+1}V_r(a_1; a_6, \dots, a_{r+1}; q, p) \\
&= \sum_{k=0}^{\infty} \frac{\theta(a_1 q^{2k}; p)}{\theta(a_1; p)} \frac{(a_1, a_6, \dots, a_{r+1}; q, p)_k}{(q, a_1q/a_6, \dots, a_1q/a_{r+1}; q, p)_k} q^k,
\end{aligned}
$$

where $\theta(a; p)$ is a theta function

$$\theta(a; p) = \prod_{i=0}^{\infty} (1 - ap^i)(1 - p^{i+1}/a), \qquad 0 < |p| < 1,$$

and $(a; q, p)_n$ is the elliptic analogue of the q-shifted factorial

$$(a; q, p)_n = \prod_{j=0}^{n-1} \theta(aq^j; p).$$

As usual,

$$(a_1, \dots, a_k; q, p)_n = (a_1; q, p)_n \dots (a_k; q, p)_n.$$

For reasons of convergence one must impose that one of the parameters a_i is of the form q^{-n} so that the above series terminates. Furthermore, to obtain non-trivial results, r must be odd and

$$a_6 \cdots a_{r+1} q = (a_1 q)^{(r-5)/2}.$$

For ordinary as well as basic hypergeometric series a vast number of summation identities are known, see e.g., [8, 18]. Unfortunately, most of these do not appear to have an elliptic analogue and to the best of my knowledge the only two summation identities for series of the type (1.1) known to date are the elliptic Jackson sum of Frenkel and Turaev [7, Theorem 5.5.2]

$$(1.2)\qquad {}_{10}V_9(a; b, c, d, e, q^{-n}; q, p) = \frac{(aq, aq/bc, aq/bd, aq/cd; q, p)_n}{(aq/b, aq/c, aq/d, aq/bcd; q, p)_n},$$

2000 *Mathematics Subject Classification.* 33D15, 33E05.
Key words and phrases. basic hypergeometric series, elliptic hypergeometric series.
Work supported by the Australian Research Council.

for $bcde = a^2q^{n+1}$, and the elliptic Gasper sum

$$
{}_{2r+8}V_{2r+7}(a; b, a/b, c_1q^{m_1}, \dots, c_rq^{m_r}, aq/c_1, \dots, aq/c_r, q^{-n}; q, p)
= \frac{(q, aq; q, p)_n}{(bq, aq/b; q, p)_n} \prod_{j=1}^{r} \frac{(c_j/b, c_jb/a; q, p)_{m_j}}{(c_j, c_j/a; q, p)_{m_j}},
$$

for $m_1, \dots, m_r$ integers such that $m_1 + \cdots + m_r = n$. This second summation was proved for $(c_j, m_j, q) \to (cq^{r-j}, n/r, q^r)$ in [24, Theorem 4.1] and in full generality by Rosengren and Schlosser in [17, Equation (1.7)].

In a recent paper [25] I stated without proof that

$$
\sum_{k=0}^{n} \frac{\theta(a^2q^{4k}; p^2)}{\theta(a^2; p^2)} \frac{(a^2, b/q; q^2, p^2)_k}{(q^2, a^2q^3/b; q^2, p^2)_k} \frac{(aq^n/b, q^{-n}; q, p)_k}{(bq^{1-n}, aq^{n+1}; q, p)_k} q^{2k}
= \frac{\theta(-aq^{2n}/b; p)}{\theta(-a/b; p)} \frac{(-a/b, aq; q, p)_n}{(-q, 1/b; q, p)_n} \frac{(1/bq; q^2, p^2)_n}{(a^2q^3/b; q^2, p^2)_n} q^n. \tag{1.3}
$$

When p tends to zero this simplifies to a bibasic summation of Nassrallah and Rahman [11, Corollary 4] (see also [8, Equation (3.10.8)]). Initially I was only able to find a rather unpleasant inductive proof, but an e-mail exchange with Vyacheslav Spiridonov prompted me to try again to find a more constructive derivation of (1.3). In this paper I will give such a proof. Interestingly, it depends crucially on the new elliptic identity

$$
{}_{12}V_{11}(ab; b, bq, b/p, bqp, aq^2/b, a^2q^{2n}, q^{-2n}; q^2, p^2)
= \frac{\theta(a; p)}{\theta(aq^{2n}; p)} \frac{(-q, aq/b; q, p)_n}{(a, -b; q, p)_n} \frac{(abq^2; q^2, p^2)_n}{(a/b; q^2, p^2)_n} q^{-n}, \tag{1.4}
$$

which provides a third example of a summable ${}_{r+1}V_r$ series.

The quasi-periodicity of the theta functions

$$
\theta(a; p) = -a\,\theta(ap; p) \tag{1.5}
$$

yields

$$
(a; q, p)_n = (-a)^n q^{\binom{n}{2}} (ap; q, p)_n. \tag{1.6}
$$

Moreover, from

$$
\lim_{p\to 0} \theta(ap; p^2) = 1 \tag{1.7}
$$

it follows that

$$
\lim_{p\to 0} (ap; q, p^2)_n = 1. \tag{1.8}
$$

Hence

$$
\lim_{p\to 0} \frac{(b/p; q^2, p^2)_k}{(aq/p; q^2, p^2)_k} = \Bigl(\frac{b}{aq}\Bigr)^k \lim_{p\to 0} \frac{(bp; q^2, p^2)_k}{(aqp; q^2, p^2)_k} = \Bigl(\frac{b}{aq}\Bigr)^k.
$$

Using standard notation for basic hypergeometric series [8] it thus follows that in the $p \to 0$ limit (1.4) becomes

$$
{}_8W_7(ab; b, bq, aq^2/b, a^2q^{2n}, q^{-2n}; q^2, bq/a)
= \frac{1-a}{1-aq^{2n}} \frac{(-q, aq/b; q)_n}{(a, -b; q)_n} \frac{(abq^2; q^2)_n}{(a/b; q^2)_n} q^{-n}.
$$

Using Watson's ${}_8\phi_7$ transformation [8, Equation (III.18)] this may be also put as

$$ {}_4\phi_3\left[\begin{matrix} b, bq, a^2q^{2n}, q^{-2n} \\ b^2, aq, aq^2 \end{matrix}; q^2, q^2\right] = \frac{1-a}{1-aq^{2n}} \frac{(-q, aq/b; q)_n}{(a, -b; q)_n} b^n, \tag{1.9} $$

an identity which follows by specializing [11, Equation (4.8)] (see also [8, Equation (3.10.14)]).

Given (1.4) the proof of (1.3) is routine, but proving (1.4) is unexpectetly difficult since its constructive proof requires (1.3)! In the next section I will therefore give a rather non-standard proof of (1.4) by specializing a recent elliptic transformation formula of Spiridonov in a singular point. The bonus of this proof is that it immediately suggests the following companion to (1.4)

$$ \begin{aligned} &{}_{12}V_{11}(ab; b, -b, bp, -b/p, aq/b, a^2q^{n+1}, q^{-n}; q, p^2) \\ &\qquad = \chi(n \text{ is even}) \frac{(q, a^2q^2/b^2; q^2, p^2)_{n/2}}{(a^2q^2, b^2q; q^2, p^2)_{n/2}} \frac{(abq; q, p^2)_n}{(aq/b; q, p^2)_n}, \end{aligned} \tag{1.10} $$

with $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$. This is the fourth example of a ${}_{r+1}V_r$ that can be summed. In the limit when p tends to zero (1.10) simplifies to

$$ \begin{aligned} &{}_8W_7(ab; b, -b, aq/b, a^2q^{n+1}, q^{-n}; q, -b/a) \\ &\qquad = \chi(n \text{ is even}) \frac{(q, a^2q^2/b^2; q^2)_{n/2}}{(a^2q^2, b^2q; q^2)_{n/2}} \frac{(abq; q)_n}{(aq/b; q)_n}. \end{aligned} $$

By Watson's ${}_8\phi_7$ transformation this can be further reduced to Andrews' terminating q-analogue of Watson's ${}_3F_2$ sum [1, Theorem 1] (see also [8, Equation (II.17)])

$$ {}_4\phi_3\left[\begin{matrix} b, -b, a^2q^{n+1}, q^{-n} \\ b^2, aq, -aq \end{matrix}; q, q\right] = \chi(n \text{ is even}) \frac{(q, a^2q^2/b^2; q^2, p)_{n/2}}{(a^2q^2, b^2q; q^2, p)_{n/2}} b^n. \tag{1.11} $$

The identities (1.4) and (1.10) together with Watson's transformation imply the ${}_4\phi_3$ sums (1.9) and (1.11). It is however also possible to rewrite (1.4) and (1.10) as two elliptic summations that yield (1.9) and (1.11) when p tends to zero without an appeal to Watson's transformation. Making the substitution $a \to ap$ in (1.4) and using the quasi-periodicities (1.5) and (1.6) yields

$$ \begin{aligned} &{}_{12}V_{11}(abp; b, bq, bp, bqp, aq^2p/b, a^2q^{2n}, q^{-2n}; q^2, p^2) \\ &\qquad = \frac{\theta(a; p)}{\theta(aq^{2n}; p)} \frac{(-q, aq/b; q, p)_n}{(a, -b; q, p)_n} \frac{(abq^2p; q^2, p^2)_n}{(ap/b; q^2, p^2)_n} b^n. \end{aligned} \tag{1.12} $$

By (1.7) and (1.8) the $p \to 0$ limit breaks the very-well-poisedness, resulting in (1.9). In much the same way, replacing $a \to ap$ in (1.10) and using (1.5) and (1.6) yields

$$ \begin{aligned} &{}_{12}V_{11}(abp; b, -b, bp, -bp, aqp/b, a^2q^{n+1}, q^{-n}; q, p^2) \\ &\qquad = \chi(n \text{ is even}) \frac{(q, a^2q^2/b^2; q^2, p^2)_{n/2}}{(a^2q^2, b^2q; q^2, p^2)_{n/2}} \frac{(abqp; q, p^2)_n}{(aqp/b; q, p^2)_n} b^n. \end{aligned} \tag{1.13} $$

When p tend to 0 this reduces to (1.11).

The results (1.12) and (1.13) show that, potentially, many more identities for series that are balanced but not very-well poised may have an elliptic analogue.

Indeed, after showing him (1.12) and (1.13), Michael Schlosser observed that making the simultaneous variable changes $(a,d,e,p)\to(ap,aqp/d,ep,p^2)$ in (1.2) gives

$$ {}_{10}V_9(ap;b,c,aqp/d,ep,q^{-n};q,p^2)=\frac{(aqp,aqp/bc,d/b,d/c;q,p^2)_n}{(aqp/b,aqp/c,d,d/bc;q,p^2)_n}, $$

for $bce=adq^n$. In the $p\to 0$ limit this results in the q-Pfaff–Saalschütz sum [8, Equation (II.12)]

$$ {}_3\phi_2\left[\begin{matrix} b,c,q^{-n} \\ d,bcq^{1-n}/d\end{matrix};q,q\right]=\frac{(d/b,d/c;q)_n}{(d,d/bc;q)_n}. $$

Probably the most important balanced summation not yet treated is Andrews' terminating q-analogue of Whipple's ${}_3F_2$ sum [1, Theorem 2] (see also [8, Equation (II.19)])

$$ {}_4\phi_3\left[\begin{matrix} b,-b,q^{n+1},q^{-n} \\ -q,c,b^2q/c\end{matrix};q,q\right]=\frac{(c/b^2;q)_n}{(c;q)_n}\frac{(cq^{-n};q^2)_n}{(cq^{-n}/b^2;q^2)_n}. \tag{1.14} $$

To obtain its elliptic analogue I will first prove the new identity

$$ \begin{aligned} &{}_{12}V_{11}(b;-b,bp,-b/p,c/b,bq/c,q^{n+1},q^{-n};q,p^2) \\ &\qquad=\frac{(bq,c/b^2;q,p^2)_n}{(q/b,c;q,p^2)_n}\frac{(cq^{-n};q^2,p^2)_n}{(cq^{-n}/b^2;q^2,p^2)_n}(-1/b)^n. \end{aligned} \tag{1.15} $$

Replacing $b\to bp$ and using (1.5) and (1.6) this implies the identity

$$ \begin{aligned} &{}_{12}V_{11}(bp;b,-b,-bp,cp/b,bpq/c,q^{n+1},q^{-n};q,p^2) \\ &\qquad=\frac{(bqp,c/b^2;q,p^2)_n}{(qp/b,c;q,p^2)_n}\frac{(cq^{-n};q^2,p^2)_n}{(cq^{-n}/b^2;q^2,p^2)_n}, \end{aligned} $$

which simplifies to (1.14) when p tends to 0 thanks to (1.7) and (1.8).

2. Proofs of (1.3), (1.4), (1.10) and (1.15)

First I will give a proof of (1.3) assuming (1.4), and a proof of (1.4) assuming (1.3). Then I will give a different proof of (1.4) based on the transformation (2.3) below.

Proof of (1.3) *based on* (1.4). When $cd=aq$ equation (1.2) simplifies to

$$ {}_8V_7(a;b,aq^n/b,q^{-n};q,p)=\delta_{n,0}, \tag{2.1} $$

with $\delta_{n,m}=\chi(n=m)$. Making the simultaneous replacements

$$ (a,b,n,q,p)\to(a^2,b/q,r,q^2,p^2), $$

then multiplying both sides by

$$ \frac{\theta(a^2q^{4r+1}/b;p^2)}{\theta(a^2q/b;p^2)}\frac{(-aq;q,p)_{2r}}{(-aq/b;q,p)_{2r}}\frac{(a^2q/b,q/b,a^2q^{2n}/b^2,q^{-2n};q^2,p^2)_r}{(q^2,a^2q^2,bq^{3-2n},a^2q^{2n+3}/b;q^2,p^2)_r}(bq^2)^r $$

and finally summing r from 0 to n yields

$$ \begin{aligned} &\sum_{r=0}^n\frac{\theta(a^2q^{4r+1}/b;p^2)}{\theta(a^2q/b;p^2)}\frac{(-aq;q,p)_{2r}}{(-aq/b;q,p)_{2r}}\frac{(a^2q/b,q/b,a^2q^{2n}/b^2,q^{-2n};q^2,p^2)_r}{(q^2,a^2q^2,bq^{3-2n},a^2q^{2n+3}/b;q^2,p^2)_r}(bq^2)^r \\ &\qquad\times{}_8V_7(a^2;b/q,a^2q^{2r+1}/b,q^{-2r};q^2,p^2)=1. \end{aligned} $$

Interchanging the order of summation and using the identity

$$\frac{(a;q,p)_{2n}}{(b;q,p)_{2n}}=\frac{(a,aq,a/p,aqp;q^2,p^2)_n}{(b,bq,b/p,bqp;q^2,p^2)_n}\Big(\frac{b}{a}\Big)^n \tag{2.2}$$

this becomes

$$\sum_{s=0}^{n}\frac{(-aq,q,p)_{2s}}{(-aq/b;q,p)_{2s}}\frac{(a^2q^3/b;q^2,p^2)_{2s}}{(a^2;q^2,p^2)_{2s}}\frac{(a^2,b/q,a^2q^{2n}/b^2,q^{-2n};q^2,p^2)_s}{(q^2,a^2q^3/b,bq^{3-2n},a^2q^{2n+3}/b;q^2,p^2)_s}\,q^{3s}$$
$$\times\,{}_{12}V_{11}(a^2q^{4s+1}/b;-aq^{2s+1},-aq^{2s+2},-aq^{2s+1}/p,-aq^{2s+2}p,$$
$$q/b,a^2q^{2n+2s}/b^2,q^{2s-2n};q^2,p^2)=1.$$

Summing the ${}_{12}V_{11}$ series by (1.4) and making some simplifications completes the proof. □

Proof of (1.4) *based on* (1.3). Replacing

$$(a,b,n,q,p)\to(a,aq^2/b^2,r,q^2,p^2)$$

in (2.1), multiplying both sides by

$$\frac{\theta(b^2q^{4r-2};p^2)}{\theta(b^2/q^2;p^2)}\frac{(b^2/q^2,b^2/aq^2;q^2,p^2)_r}{(q^2,aq^2;q^2,p^2)_r}\frac{(-aq^n/b,q^{-n};q,p)_r}{(b^2q^{-n}/a,-bq^n;q,p)_r}\,q^{2r}$$

and summing r from 0 to n yields

$$\sum_{r=0}^{n}\frac{\theta(b^2q^{4r-2};p^2)}{\theta(b^2/q^2;p^2)}\frac{(b^2/q^2,b^2/aq^2;q^2,p^2)_r}{(q^2,aq^2;q^2,p^2)_r}\frac{(-aq^n/b,q^{-n};q,p)_r}{(b^2q^{-n}/a,-bq^n;q,p)_r}\,q^{2r}$$
$$\times\,{}_8V_7(a;aq^2/b^2,b^2q^{2r-2},q^{-2r};q^2,p^2)=1.$$

A change in the order of summation leads to

$$\sum_{s=0}^{n}\frac{\theta(b^2q^{4s-2};p^2)}{\theta(b^2/q^2;p^2)}\frac{(b^2/q^2;q^2,p^2)_{2s}}{(a;q^2,p^2)_{2s}}\frac{(a,aq^2/b^2;q^2,p^2)_s}{(q^2,b^2;q^2,p^2)_s}\frac{(-aq^n/b,q^{-n};q,p)_s}{(b^2q^{-n}/a,-bq^n;q,p)_s}$$
$$\times\Big(\frac{b^2}{a}\Big)^s\sum_{r=0}^{n-s}\frac{\theta(b^2q^{4r+4s-2};p^2)}{\theta(b^2q^{4s-2};p^2)}\frac{(b^2q^{4s-2},b^2/aq^2;q^2,p^2)_r}{(q^2,aq^{4s+2};q^2,p^2)_r}$$
$$\times\frac{(-aq^{n+s}/b,q^{s-n};q,p)_r}{(b^2q^{s-n}/a,-bq^{n+s};q,p)_r}\,q^{2r}=1.$$

The sum over r can be performed by (1.3) giving

$$\sum_{s=0}^{n}\frac{\theta(aq^{4s};p^2)}{\theta(a;p^2)}\frac{(b;q,p)_{2s}}{(aq/b;q,p)_{2s}}\frac{(a,aq^2/b^2,a^2q^{2n}/b^2,q^{-2n};q^2,p^2)_s}{(q^2,b^2,b^2q^{2-2n}/a,aq^{2n+2};q^2,p^2)_s}\Big(\frac{b^2q}{a}\Big)^s$$
$$=q^{-n}\frac{\theta(a/b;p)}{\theta(aq^{2n}/b;p)}\frac{(-q,aq/b^2;q,p)_n}{(a/b,-b;q,p)_n}\frac{(aq^2;q^2,p^2)_n}{(a/b^2;q^2,p^2)_n}.$$

Once more using (2.2) and replacing a by ab completes the proof. □

Proof of (1.4). To give a proof of (1.4) that does not rely on (1.3) I need the following transformation formula of Spiridonov [21, Theorem 5.1] (see also [25, Theorem 4.1]):

$$(2.3)\quad {}_{14}V_{13}(a;a^2q/m,b^{1/2},-b^{1/2},c^{1/2},-c^{1/2},k^{1/2}q^n,-k^{1/2}q^n,q^{-n},-q^{-n};q,p)$$
$$=\frac{(a^2q^2,k/m,mq^2/b,mq^2/c;q^2,p^2)_n}{(mq^2,k/a^2,a^2q^2/b,a^2q^2/c;q^2,p^2)_n}$$
$$\times\,{}_{14}V_{13}(m;a^2q^2/m,d,dq,d/p,dqp,b,c,kq^{2n},q^{-2n};q^2,p^2),$$

for $m=bck/a^2q^2$ and $d=-m/a$. When p tends to 0 this becomes

$$(2.4)\quad {}_{12}W_{11}(a;a^2q/m,b^{1/2},-b^{1/2},c^{1/2},-c^{1/2},k^{1/2}q^n,-k^{1/2}q^n,q^{-n},-q^{-n};q,q)$$
$$=\frac{(a^2q^2,k/m,mq^2/b,mq^2/c;q^2)_n}{(mq^2,k/a^2,a^2q^2/b,a^2q^2/c;q^2)_n}$$
$$\times\,{}_{10}W_9(m;a^2q^2/m,d,dq,b,c,kq^{2n},q^{-2n};q^2,mq/a^2)$$

which is equivalent to a bibasic transformation of Nassrallah and Rahman [11, Equation (4.14)] (see also [8, Equation (3.10.15)]). In the above representation (2.4) has been rediscovered very recently in [2, Equation (4.9)].

To now prove (1.4) I observe that the ${}_{14}V_{13}$ series on the left side of (2.3) as well as the prefactor on the right side of (2.3) are singular for $k=a^2$. Multiplying both sides by $(k/a^2;q^2,p^2)_n$ and observing that for $0\le r\le n$

$$\lim_{k\to a^2}\frac{(k/a^2;q^2,p^2)_n}{(a^2q^{2-2n}/k;q^2,p^2)_r}=(-1)^nq^{n^2-n}\delta_{n,r},$$

it follows that in the limit when k tends to a^2 only the term with $r=n$ survives in the sum on the left (with r being the summation index of the ${}_{14}V_{13}$ series). As a result

$${}_{12}V_{11}(m;a^2q^2/m,d,dq,d/p,dqp,a^2q^{2n},q^{-2n};q^2,p^2)$$
$$=q^{-n}\frac{\theta(-a;p)}{\theta(-aq^{2n};p)}\frac{(-q,a^2q/m;q,p)_n}{(-a,m/a;q,p)_n}\frac{(mq^2;q^2,p^2)_n}{(a^2/m;q^2,p^2)_n},$$

with $m=bc/q^2$ and $d=-m/a$. Since the only dependence on b and c is through the definition of m, the equation $m=bc/q^2$ is superfluous, and the above is true with a and m arbitrary indeterminates. Making the simultaneous changes $m\to ab$ and $a\to -a$ yields (1.4). □

Proof of (1.10). As mentioned in the introduction, the above proof of (1.4) immediately suggests (1.10) by virtue of the fact that (2.3) has the companion [25, Theorem 4.2]

$$(2.5)\quad {}_{14}V_{13}(a;a^2/m^2,b,bq,c,cq,kq^n,kq^{n+1},q^{-n},q^{1-n};q^2,p)$$
$$=\frac{(aq,k/m,mq/b,mq/c;q,p)_n}{(mq,k/a,aq/b,aq/c;q,p)_n}$$
$$\times\,{}_{14}V_{13}(m;a/m,d,-d,dp^{1/2},-d/p^{1/2},b,c,kq^n,q^{-n};q,p),$$

for $m=bck/aq$ and $d=m(q/a)^{1/2}$. In the $p\to 0$ limit this gives

$$(2.6)\quad {}_{12}W_{11}(a;a^2/m^2,b,bq,c,cq,kq^n,kq^{n+1},q^{-n},q^{1-n};q^2,q^2)$$
$$=\frac{(aq,k/m,mq/b,mq/c;q)_n}{(mq,k/a,aq/b,aq/c;q)_n}\,{}_{10}W_9(m;a/m,d,-d,b,c,kq^n,q^{-n};q,-mq/a)$$

due to Rahman and Verma [12, Equation (7.8)] (see also [2, Equation (3.13)]).

This time the singularity to be exploited occurs for $k = a$. Multiplying both sides of (2.5) by $(k/a;q,p)_n$ and observing that for $0 \le 2r \le n$

$$\lim_{k\to a} \frac{(k/a;q,p)_n}{(aq^{1-n}/k;q^2,p^2)_{2r}} = q^{\binom{n}{2}}\delta_{n,2r},$$

it follows that in the $k \to a$ limit only the term with $2r = n$ survives in the sum on the left (with r being the summation index of the ${}_{14}V_{13}$ series). Hence

$$\begin{aligned}&{}_{12}V_{11}(m;a/m,d,-d,dp^{1/2},-d/p^{1/2},aq^n,q^{-n};q,p)\\ &\qquad\qquad = \chi(n\text{ even})\frac{(a,a^2/m^2;q^2,p)_{n/2}}{(q^2,m^2q^2/a;q^2,p)_{n/2}}\frac{(q,mq;q,p)_n}{(a,a/m;q,p)_n}\end{aligned}$$

with $m = bc/q$ and $d = m(q/a)^{1/2}$. Again the dependence on b and c is only through the definition of m, so that the above is true for arbitrary a and m. Making the simultaneous changes $m \to ab$, $a \to a^2q$ and $p \to p^2$ yields (1.10). □

Proof of (1.15). Making the simultaneous substitutions

$$(a,b,c,d,e,f,g,p) \to (b,c/b,bq/c,q^{n+1},-b,bp,-b/p,p^2)$$

in the elliptic analogue of Bailey's ${}_{10}\phi_9$ transformation [7, Theorem 5.5.1]

$$\begin{aligned}&{}_{12}V_{11}(a;b,c,d,e,f,g,q^{-n};q,p)\\ &\quad = \frac{(aq,aq/ef,aq/fg,aq/eg;q,p)_n}{(aq/e,aq/f,aq/g,aq/efg;q,p)_n}\,{}_{12}V_{11}(\lambda;\lambda b/a,\lambda c/a,\lambda d/a,e,f,g,q^{-n};q,p)\end{aligned}$$

for $bcdefg = a^3q^{n+2}$ and $\lambda = a^2q/bcd$, (1.15) can be transformed into

$$\begin{aligned}&{}_{12}V_{11}(b^2q^{-n-1};b,-b,bp,-b/p,cq^{-n-1},b^2q^{-n}/c,q^{-n};q,p^2)\\ &\qquad\qquad = \frac{(q/b^2,c/b^2;q,p^2)_n}{(q,c;q,p^2)_n}\frac{(q^2,cq^{-n};q^2,p^2)_n}{(q^2/b^2,cq^{-n}/b^2;q^2,p^2)_n}.\end{aligned} \tag{2.7}$$

Here the right-hand side has been simplified using

$$\frac{(a,-a,a/p,-ap;q,p^2)_n}{(b,-b,bp,-b/p;q,p^2)_n} = \frac{(a^2;q^2,p^2)_n}{(b^2;q^2,p^2)_n}\Big(-\frac{a}{b}\Big)^n$$

with $a \to q$ and $b \to q/b$. When viewed as functions of c it is easy to see from (1.6) that both sides of (2.7) satisfy $f(c) = f(cp^2)$. Consequently it is enough to give a proof for $c = q^{n-m+1}$ with m an integer such that $m \ge 2n+1$. But this is nothing but (1.10) with $n \to m$ and $a \to bq^{-n-1}$. □

Acknowledgements. I thank Vyacheslav Spiridonov for prompting me to look for a proof of (1.3) beyond induction, and Michael Schlosser for helpful discussions. I thank George Gasper and Mizan Rahman for pointing out (1.9), (2.4) and (2.6) in the literature.

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