A GENERALIZATION OF THE FARKAS AND KRA PARTITION THEOREM FOR MODULUS 7

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Dedicated to George Andrews on the occasion of his 65th birthday

Abstract. We prove generalizations of some partition theorems of Farkas and Kra.

1. The Farkas and Kra partition theorem and its generalization

In a recent talk given at the University of Melbourne, Hershel Farkas asked for a(nother) proof of the following elegant partition theorem.

Theorem 1.1 (Farkas and Kra [2],[3, Chapter 7]). *Consider the positive integers such that multiples of* 7 *occur in two copies, say* $7k$ *and* $\overline{7k}$. *Let* $\mathcal{E}(n)$ *be the number of partitions of the even integer* $2n$ *into distinct even parts and let* $\mathcal{O}(n)$ *be the number of partitions of the odd integer* $2n+1$ *into distinct odd parts. Then* $\mathcal{E}(n)=\mathcal{O}(n)$.

For example, $\mathcal{E}(9)=\mathcal{O}(9)=9$, with the following admissible partitions of 18 and 19:

$$(18),(16,2),(14,4),(\overline{14},4),(12,6),(12,4,2),(10,8),(10,6,2),(8,6,4)$$

and

$$(19),(15,3,1),(13,5,1),(11,7,1),(11,\overline{7},1),\\ (11,5,3),(9,7,3),(9,\overline{7},3),(7,\overline{7},5).$$

Farkas and Kra proved their theorem as follows. It is not difficult to see that the generating function identity underlying Theorem 1.1 is

$$(1)\quad (-q;q^2)_\infty(-q^7;q^{14})_\infty-(q;q^2)_\infty(q^7;q^{14})_\infty \\ =2q(-q^2;q^2)_\infty(-q^{14};q^{14})_\infty,$$

where $(a;q)_\infty=\prod_{n\geq 0}(1-aq^n)$. Indeed, the right-hand side without the factor $2q$ is the generating function of partitions into even parts (see e.g., [1]). The first term on the left gives the generating function

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of partitions into odd parts. Since an even number of odd parts corresponds to a partition of an even number, all even powers of q on the left must be suppressed. This is achieved by antisymmetrizing, i.e., by subtracting the same term with q replaced by $-q$ and dividing both terms by 2. Putting the 2 on the right and also putting in an extra factor q to account for the fact that we are equating (the number of) partitions of $2n$ with (the number of) partitions of $2n+1$ leads to (1).

In order to prove (1), Farkas and Kra resorted to the theory of theta functions. Adopting the notation of [4], equation (1) corresponds to the following identity for theta Nullwerte:

$$\sqrt{\vartheta_2(0|\tau)\vartheta_2(0|7\tau)}+\sqrt{\vartheta_4(0|\tau)\vartheta_4(0|7\tau)}=\sqrt{\vartheta_3(0|\tau)\vartheta_3(0|7\tau)}, \tag{2}$$

where q has been identified with $\exp(2\pi\mathrm{i}\tau/7)$.

Proving (2) is not entirely trivial [2],[3, Chapter 4], and when Farkas asked for a different proof of Theorem 1.1 he was a referring to a proof without the use of theta functions. Ideally one would of course like to establish a bijection between the partitions counted by $\mathcal{E}(n)$ and $\mathcal{O}(n)$, but that seems too difficult a problem. What we will do instead is show that (1) is the specialization of a more general identity that makes no reference to the modulus 7 (or 14). This immediately implies analogues of Theorem 1.1 for other moduli. For instance, we may claim the following.

Theorem 1.2. *Consider the positive integers such that numbers congruent to* $\pm 6 \pmod{16}$ *are forbidden and such that multiples of* 8 *occur in two copies. With* $\mathcal{E}(n)$ *and* $\mathcal{O}(n)$ *defined as in Theorem 1.1 there holds* $\mathcal{E}(n)=\mathcal{O}(n)$.

Although the description of the even integers is slightly more complicated than that of Theorem 1.1 it is to be noted that we are dealing with just the ordinary odds.

As an example of Theorem 1.2 let us list the 8 admissible partitions of 20 and of 21:

$$(20),(18,2),(16,4),(\overline{16},4),(14,4,2),(12,8),(12,\overline{8}),(8,\overline{8},4)$$

and

$$(21),(17,3,1),(15,5,1),(13,7,1),(13,5,3),(11,7,3),(11,9,1),(9,7,5).$$

More generally the following statement is true.

Theorem 1.3. *Let* α *and* β *be even positive integers such that* $\alpha<\beta$, *and let* γ *be an odd positive integer. Fix an integer* $m\geq\alpha+\beta+2\gamma+1$. *Consider the positive integers in which multiples of* $2m$ *occur in two copies,* $2m$ *and* $\overline{2m}$. *Let* $\mathcal{E}_{\alpha,\beta,\gamma;m}(n)$ *be the number of partitions of* $2n$ *with parts congruent to* $0,\overline{0},\pm\alpha,\pm\beta,\pm(\alpha+\beta+2\gamma) \pmod{2m}$ *and let*

$\mathcal{O}_{\alpha,\beta,\gamma;m}(n)$ *be the number of partitions of* $2n+\gamma$ *with parts congruent to* $\pm\gamma, \pm(\alpha+\gamma), \pm(\beta+\gamma), \pm(\alpha+\beta+\gamma) \pmod{2m}$*. Then* $\mathcal{E}_{\alpha,\beta,\gamma;m}(n) = \mathcal{O}_{\alpha,\beta,\gamma;m}(n)$*.*

Here "the parts congruent to $\overline{0} \pmod{2m}$" refers to the parts $\overline{2m}$, $\overline{4m}$, etc.

For example, for $(\alpha,\beta,\gamma) = (2,4,3)$ and $m = 13$ we have 11 partitions of 40;

$$(40), (38,2), (28,12), (26,14), (\overline{26},14), (26,12,2), (\overline{26},12,2),$$
$$(24,14,2), (24,12,4), (22,14,4), (22,12,4,2)$$

and 11 partitions of 43;

$$(43), (35,5,3), (33,7,3), (31,9,3), (31,7,5), (26,17), (26,9,5,3),$$
$$(23,17,3), (21,19,3), (21,17,5), (19,17,7).$$

Hence $\mathcal{E}_{2,4,3;13}(20) = \mathcal{O}_{2,4,3;13}(20)$.

As will be clear from the proof of Theorem 1.3 the conditions $\alpha < \beta$ and $m \geq \alpha+\beta+2\gamma+1$ are more restrictive than necessary and may be replaced by the conditions that the (unordered) sequences $\alpha, 2m-\alpha, \beta, 2m-\beta, \alpha+\beta+2\gamma, 2m-\alpha-\beta-2\gamma$ and $\gamma, 2m-\gamma, \alpha+\gamma, 2m-\alpha-\gamma, \beta+\gamma, 2m-\beta-\gamma$ consist of distinct positive integers. Moreover, if we were to choose α, β and γ such that some of the above integers would coincide, then the theorem is still correct as long as we introduce different copies of these numbers (and those in the same congruence class modulo $2m$). Hence the Farkas and Kra theorem corresponds to $(\alpha,\beta,\gamma) = (2,4,1)$ and $m = 7$. Then $\alpha+\beta+\gamma = 2m-\alpha-\beta-\gamma = 7$ requiring the numbers $7 \pmod{14}$ and $\overline{7} \pmod{14}$. Since we already have two copies of multiples of 14 this implies that multiples of 7 occur in two copies in accordance with Theorem 1.1.

Similarly, for $(\alpha,\beta,\gamma) = (2,4,1)$ and $m = 5$ we obtain the statement that the number of partitions of $2n$ into even parts is equinumerous to the number of partitions of $2n+1$ into odd parts provided all numbers unequal to $\pm 1 \pmod 5$ occur in two copies.

2. Proof of Theorem 1.3

Repeating the analysis that led from Theorem 1.1 to the identity (1), it is not hard to see that the generating function version of Theorem 1.3

is

$$\begin{aligned}&(-q^{\gamma},-q^{\alpha+\gamma},-q^{\beta+\gamma},-q^{\alpha+\beta+\gamma},\\&\qquad -q^{2m-\gamma},-q^{2m-\alpha-\gamma},-q^{2m-\beta-\gamma},-q^{2m-\alpha-\beta-\gamma};q^{2m})_\infty\\&-(q^{\gamma},q^{\alpha+\gamma},q^{\beta+\gamma},q^{\alpha+\beta+\gamma},q^{2m-\gamma},q^{2m-\alpha-\gamma},q^{2m-\beta-\gamma},q^{2m-\alpha-\beta-\gamma};q^{2m})_\infty\\&\qquad=2q^{\gamma}(-q^{\alpha},-q^{\beta},-q^{\alpha+\beta+2\gamma},\\&\qquad\qquad -q^{2m-\alpha},-q^{2m-\beta},-q^{2m-\alpha-\beta-2\gamma},-q^{2m},-q^{2m};q^{2m})_\infty,\end{aligned}$$

where $(a_1,\dots,a_k;q)_\infty=(a_1;q)_\infty\cdots(a_k;q)_\infty$. The conditions imposed on the theorem simply ensure that no exponents other than $2m$ occur twice so that only multiples of $2m$ occur in two different copies. However, as a mere q-series identity no restrictions whatsoever need to be imposed on α,β,γ and m, and the above is simply a consequence of

$$\begin{aligned}(3)\quad &(-c,-ac,-bc,-abc,-q/c,-q/ac,-q/bc,-q/abc;q)_\infty\\&\qquad-(c,ac,bc,abc,q/c,q/ac,q/bc,q/abc;q)_\infty\\&\qquad=2c(-a,-b,-abc^2,-q/a,-q/b,-q/abc^2,-q,-q;q)_\infty.\end{aligned}$$

Replacing $q\to q^{14}$ and then letting $(a,b,c)\to(q^2,q^4,q)$ yields (1) since

$$(-q,-q^3,-q^5,-q^7,-q^9,-q^{11},-q^{13};q^{14})_\infty=(-q;q^2)_\infty$$
$$(-q^2,-q^4,-q^6,-q^8,-q^{10},-q^{12},-q^{14};q^{14})_\infty=(-q^2;q^2)_\infty.$$

Proving identities like (3) is elementary and below we give three proofs — one analytic, one using q-series and one combinatorial.

First proof of (3). Let us view the left and right-hand sides as functions of a and write $L(a)$ and $R(a)$, respectively. Define $f(a)=L(a)/R(a)$. Since $R(aq)=R(a)/a^2bc^2$ and $L(aq)=L(a)/a^2bc^2$ it follows that $f(aq)=f(a)$. Possible poles of f are given by the zeros of $R(a)$, i.e., by $a=-q^n$ and $a=-q^n/bc^2$ with n an integer. But

$$\begin{aligned}L(-q^n)&=(-c,cq^n,-bc,bcq^n,-q/c,q^{1-n}/c,-q/bc,q^{1-n}/bc;q)_\infty\\&\qquad-(c,-cq^n,bc,-bcq^n,q/c,-q^{1-n}/c,q/bc,-q^{1-n}/bc;q)_\infty\\&=(bc^2)^{-n}q^{-2\binom{n}{2}}\big[(-c,c,-bc,bc,-q/c,q/c,-q/bc,q/bc;q)_\infty\\&\qquad-(c,-c,bc,-bc,q/c,-q/c,q/bc,-q/bc;q)_\infty\big]\\&=0\end{aligned}$$

by $(aq^n,q^{1-n}/a;q)_\infty=(-1)^n a^{-n}q^{-\binom{n}{2}}(a,q/a;q)_\infty$. In much the same way one finds that $L(-q^n/bc^2)=0$ so that the poles of f have zero residue. Hence f is an entire and bounded function and must be constant thanks to Liouville's theorem. All that remains is to show this constant is one, or, equivalently, that (3) is true for an appropriately chosen value of a. Taking $a = 1/c$ the second term on the left vanishes and we get

$$\begin{aligned}(-c, -1, -bc, -b, -q/c, -q, -q/bc, -q/b; q)_\infty \\ = 2c(-1/c, -b, -bc, -cq, -q/b, -q/bc, -q, -q; q)_\infty\end{aligned}$$

which is true since $(a; q)_\infty = (1 - a)(aq; q)_\infty$. □

Second proof of (3). We use the Jacobi triple-product identity

$$\sum_{n=-\infty}^{\infty} z^n q^{\binom{n}{2}} = (-z, -q/z, q; q)_\infty \tag{4}$$

to expand both sides of (3) as multiple series. Equating the coefficients of $a^k b^l c^{2m+1}$ leads to

$$\sum_{n=-\infty}^{\infty} q^{\binom{2m+1+n-k-l}{2}+\binom{k-n}{2}+\binom{l-n}{2}+\binom{n}{2}} = (-q, -q, q; q)_\infty q^{\binom{k-m}{2}+\binom{l-m}{2}+\binom{m}{2}}.$$

After some simplifications of the exponents of q this is just

$$\sum_{n=-\infty}^{\infty} q^{\binom{i-2n}{2}} = (-q, -q, q; q)_\infty,$$

where we have replaced $k+l-m$ by i. Making the change $n \to i/2 - n$ when i is even and $n \to n + (i-1)/2$ when n is odd, and using $\binom{-m}{2} = \binom{m+1}{2}$ in the latter case, it follows that the sum on the left is noting but

$$\sum_{n=-\infty}^{\infty} q^{\binom{2n}{2}}$$

independent of i. By the Jacobi triple-product identity with $q \to q^4$ and $z \to q$ this yields $(-q, -q^3, q^4; q^4)_\infty = (-q, -q^2, q^2; q^2)_\infty = (-q, -q, q; q)_\infty$ as desired. □

Third proof of (3). The third and final proof of (3), which employs a mapping used by Wright to bijectively prove the Jacobi triple-product identity (4), was suggested to us by the anonymous referee. In fact, it let us to discover that (3) is the $d = 1$ instance of

$$\begin{aligned}&d(-c, -q/c, -ac, -q/ac, -bc, -q/bc, -abcd^2, -q/abcd^2; q)_\infty \\ &\quad - d(c, q/c, ac, q/ac, bc, q/bc, abcd^2, q/abcd^2; q)_\infty \\ &= c(-d, -q/d, -ad, -q/ad, -bd, -q/bd, -abc^2d, -q/abc^2d; q)_\infty \\ &\quad - c(d, q/d, ad, q/ad, bd, q/bd, abc^2d, q/abc^2d; q)_\infty.\end{aligned} \tag{5}$$

Interestingly, the previous proof of (3) based on the triple-product identity simplifies when one considers this more general identity, and the reader will have no trouble verifying that the coefficients of $a^k b^l c^{2m+1} d^{2n+1}$ on the left and right sides of (5) coincide.

Let $\mathscr{P}$ be the set of 'ordinary' partitions, i.e., $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathscr{P}$ if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$, $\mathscr{D}_0$ be the set of partitions with distinct non-negative parts, i.e., $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathscr{D}_0$ if $\lambda_1 > \lambda_2 > \cdots > \lambda_r \geq 0$, and $\mathscr{D}$ be the set of partitions with distinct positive parts. The weight $|\lambda|$ and length $\ell(\lambda)$ of a partition λ is the sum of the parts and the number of parts, respectively. The unique partition (in $\mathscr{D}$, $\mathscr{D}_0$ and $\mathscr{P}$) of weight and length zero will be denoted by $\emptyset$. For example, $\pi = (6,3,1,0) \in \mathscr{D}_0$ and $\omega = (6,3,1) \in \mathscr{D}$, with $|\pi| = |\omega| = 10$, $\ell(\pi) = 4$ and $\ell(\omega) = 3$.

We begin the proof by recalling the standard fact [1] that the coefficient of $a^m q^N$ in $(-a, -q/a; q)_\infty$ is the number of partition pairs (λ, μ) such that $\lambda \in \mathscr{D}_0$, $\mu \in \mathscr{D}$, $|\lambda| + |\mu| = N$ and $\ell(\lambda) - \ell(\mu) = m$. Hence the coefficient of $a^k b^l c^{2m+1} q^N$ in

$$\begin{aligned}(6)\quad \big[(-c, -q/c, -ac, -q/ac, -bc, -q/bc, -abc, -q/abc; q)_\infty \\ - (c, q/c, ac, q/ac, bc, q/bc, abc, q/abc; q)_\infty\big]/2\end{aligned}$$

is the cardinality of the set $\mathscr{S}_{k,l,m}(N)$ whose elements consist of four partition pairs $((\lambda^{(1)}, \mu^{(1)}), \dots, (\lambda^{(4)}, \mu^{(4)}))$ such that $\lambda^{(i)} \in \mathscr{D}_0$, $\mu^{(i)} \in \mathscr{D}$, $\sum_i (|\lambda^{(i)}| + |\mu^{(i)}|) = N$, $d_2 + d_4 = k$, $d_3 + d_4 = l$ and $d_1 + d_2 + d_3 + d_4 = 2m+1$, where $d_i = \ell(\lambda^{(i)}) - \ell(\mu^{(i)})$. (The role of the second term in (6) is merely to suppress the even powers of c.) For example, the five elements of $\mathscr{S}_{2,1,0}(2)$ are

(7)

1: $((\emptyset,(2)),((0),\emptyset),(\emptyset,\emptyset),((0),\emptyset))$, 2: $((\emptyset,(1)),((0),\emptyset),(\emptyset,\emptyset),((1),\emptyset))$,
3: $((\emptyset,(1)),((1),\emptyset),(\emptyset,\emptyset),((0),\emptyset))$, 4: $((\emptyset,(1)),((0),\emptyset),((0),(1)),((0),\emptyset))$,
5: $((\emptyset,\emptyset),(\emptyset,\emptyset),(\emptyset,(1)),((1,0),\emptyset))$.

Similar considerations show that the coefficient of $a^k b^l c^{2m+1} q^N$ in

$$c(-q, -q, -a, -q/a, -b, -q/b, -abc^2, -q/abc^2; q)_\infty$$

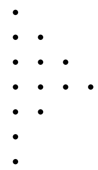
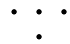
is the cardinality of the set $\mathscr{T}_{k,l,m}(N)$ whose elements consist of four partition pairs $((\lambda^{(1)}, \mu^{(1)}), \dots, (\lambda^{(4)}, \mu^{(4)}))$ such that $\lambda^{(i)} \in \mathscr{D}_0$ $(i \geq 2)$, $\lambda^{(1)}, \mu^{(i)} \in \mathscr{D}$, $\sum_i (|\lambda^{(i)}| + |\mu^{(i)}|) = N$, $d_2 + d_4 = k$, $d_3 + d_4 = l$ and $d_4 = m$. For example, the five elements of $\mathscr{T}_{2,1,0}(2)$ are

(8)

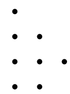
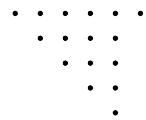
1: $((\emptyset,(1)),((1,0),\emptyset),((0),\emptyset),(\emptyset,\emptyset))$, 2: $((\emptyset,\emptyset),((1,0),\emptyset),((0),\emptyset),((0),(1)))$,
3: $((\emptyset,\emptyset),((2,0),\emptyset),((0),\emptyset),(\emptyset,\emptyset))$, 4: $((\emptyset,\emptyset),((1,0),\emptyset),((1),\emptyset),(\emptyset,\emptyset))$,
5: $(((1),\emptyset),((1,0),\emptyset),((0),\emptyset),(\emptyset,\emptyset))$.

Key in establishing a bijection between $\mathscr{S}_{k,l,m}(N)$ and $\mathscr{T}_{k,l,m}(N)$ is the following mapping of Wright [5]. Let (λ, μ) be a partition pair such that $\lambda \in \mathscr{D}_0$ and $\mu \in \mathscr{D}$ and such that $\ell(\lambda) - \ell(\mu) = d$. Draw a diagram

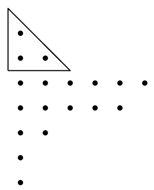
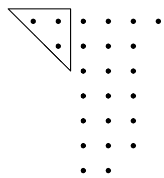
$G = G(\lambda)$ consisting of $\ell(\lambda)$ columns of nodes such that the ith column contains λ_i nodes and such that the $(i+1)$th column is displaced one unit to the right and unit down relative to the ith column. Similarly, draw a diagram $H = H(\mu)$ consisting of $\ell(\mu)$ rows of nodes such that the ith row contains μ_i nodes and such that the $(i+1)$th row is displaced one unit to the right and one unit down relative to the ith row. For example if $\lambda^{(1)} = (7,4,2,1,0)$ and $\mu^{(1)} = (3,1)$ then

$G^{(1)}$:  and $H^{(1)}$: 

and if $\lambda^{(2)} = (4,3,1)$ and $\mu^{(2)} = (6,4,3,2,1)$ then

$G^{(2)}$:  and $H^{(2)}$: 

Next glue together the sides of slope -1 of G and H to form a diagram K as follows. If $d > 0$ concatenate the top-row of H and the dth row of G. If $d \le 0$ concatenate the first column of G and the $(1-d)$th column of μ. For the two examples above the respective graphs K are

$K^{(1)}$:  and $K^{(2)}$: 

Clearly, K corresponds to a partition $\pi \in \mathscr{P}$ of weight $|\lambda|+|\mu|-\binom{d}{2}$ and a triangle of $\binom{d}{2}$ nodes, and in the running examples $\pi^{(1)} = (6,5,2,1,1)$ and $\pi^{(2)} = (4,3^5,2)$, respectively.

Conversely, given a partition $\pi \in \mathscr{P}$ and an integer d we can form a diagram K by adding a triangle of $\binom{d}{2}$ nodes to the diagram of π. When $d > 0$ this triangle must be placed on top of the diagram of π (and be left-aligned), and when $d \le 0$ it must be placed immediately to the left of π (and be top-aligned). Then K can be dissected into two diagrams G and H corresponding to a partition pair (λ, μ) such that $\lambda \in \mathscr{D}_0$ and $\mu \in \mathscr{D}$, $|\lambda| + |\mu| = |\pi| + \binom{d}{2}$ and $\ell(\lambda) - \ell(\mu) = d$ in the obvious way. In particular we note that (i) when $d > 0$ the first d

columns of K belong entirely to G (ii) when $d \le 0$ the top $(1-d)$ rows of K belongs entirely to H, (iii) after reading off the non-zero parts of λ and μ from G and H, the condition $\ell(\lambda)-\ell(\mu)=d$ determines whether λ should have an additional part equal to zero.

Let us now use Wright's map to set up a bijection between $\mathscr{S}_{k,l,m}(N)$ and $\mathscr{T}_{k,l,m}(N)$. In fact we will do a bit more and as intermediate step we will show bijectively that

$$\mathscr{S}_{k,l,m,n}(N)=\mathscr{S}_{k,l,n,m}(N),$$

with $\mathscr{S}_{k,l,m,n}(N)$ defined as the subset of $\mathscr{S}_{k,l,m}(N)$ whose elements satisfy the additional restriction $d_4=\ell(\lambda^{(4)})-\ell(\mu^{(4)})=n$. This result readily leads to the q-series identity (5).

Let $s=((\lambda^{(1)},\mu^{(1)}),\dots,(\lambda^{(4)},\mu^{(4)}))$ be an element of $\mathscr{S}_{k,l,m,n}(N)$. Hence $d_1=2m+1+n-k-l$, $d_2=k-n$, $d_3=l-n$ and $d_4=n$. Now apply Wright's map to each of the four partition pairs to obtain four ordinary partitions and four triangles indexed by $d_1,\dots,d_4$ containing a total of

$$\sum_{i=1}^{4}\binom{d_i}{2}=\binom{2m+1+n-k-l}{2}+\binom{k-n}{2}+\binom{l-n}{2}+\binom{n}{2}$$

nodes. Defining $d_1'=2n+1+m-k-l$, $d_2'=k-m$, $d_3'=l-m$ and $d_4'=m$ it is easily checked that

$$\sum_{i=1}^{4}\binom{d_i}{2}=\sum_{i=1}^{4}\binom{d_i'}{2}.$$

We may therefore redistribute the nodes of the triangles indexed by $d_1,\dots,d_4$ to form four new triangles indexed by $d_1',\dots,d_4'$. Then applying the inverse of Wright's map yields $s'=((\pi^{(1)},\omega^{(1)}),\dots,(\pi^{(4)},\omega^{(4)}))\in\mathscr{S}_{k,l,n,m}(N)$.

For example, if we take

$$\begin{aligned}s=(((9,6,4,0),(2)),((2),(4,2,1)),&\\ ((2,0),(3,2)),((4,2,0),(2)))&\in\mathscr{S}_{0,2,1,2}(45)\end{aligned}$$

then $(d_1,d_2,d_3,d_4)=(3,-2,0,2)$ and $(d_1',d_2',d_3',d_4')=(4,-1,1,1)$. Hence

$$\begin{gathered}((9,6,4,0),(2))\to(5,3^3,2,1^2)\to((10,7,5,1,0),(1))\\ ((2),(4,2,1))\to(2,1^4)\to((3),(3,1))\\ ((2,0),(3,2))\to(3^2,1)\to((3,1,0),(2,1))\\ ((4,2,0),(2)))\to(4,2,1)\to((3,1),(3))\end{gathered}$$

so that

$$s' = (((10,7,5,1,0),(1)),((3),(3,1)),\\ ((3,1,0),(2,1)),((3,1),(3))) \in \mathscr{S}_{0,2,2,1}(45).$$

To complete the bijection we map $s' \in \mathscr{S}_{k,l,n,m}(N)$ to an element $s'' \in \mathscr{T}_{k,l,m}(N)$ by removing the part 0 (if present) of the first partition in the first partition pair (this partition is denoted $\pi^{(1)}$ above) of s'. Hence in our example

$$s'' = (((10,7,5,1),(1)),((3),(3,1)),\\ ((3,1,0),(2,1)),((3,1),(3))) \in \mathscr{T}_{0,2,1}(45).$$

Conversely, given an element $s'' = ((\lambda^{(1)},\mu^{(1)}),\dots,(\lambda^{(4)},\mu^{(4)})) \in \mathscr{T}_{k,l,m}(N)$ we compute $\sigma \in \{0,1\}$ as $\sigma \equiv d_1+d_2+d_3+d_4 \pmod 2$. If $\sigma = 0$ we add a part 0 to $\lambda^{(1)}$ and if $\sigma = 1$ we leave $\lambda^{(1)}$ unchanged. This yields an element $s' \in \mathscr{T}_{k,l,n,m}(N)$ (with $d'_1 = d_1 + 1 - \sigma$, $d'_2 = d_2$, $d'_3 = d_3$ and $d'_4 = d_4$) where n is fixed by $2n+1 = d'_1+d'_2+d'_3+d'_4$.

For s'' given in the example, $(d_1,d_2,d_3,d_4) = (3,-1,1,1)$ so that $\sigma = 0$. Hence $(10,7,5,1) \to (10,7,5,1,0)$, $(d'_1,d'_2,d'_3,d'_4) = (4,-1,1,1)$ and $n = 2$.

As a further example the reader may wish to check that the proposed bijection respects the labelling of the elements of $\mathscr{S}_{2,1,0}(2)$ and $\mathscr{T}_{2,1,0}(2)$ in (7) and (8). ☐

3. Discussion

Farkas and Kra [2],[3, Chapter 7] proved two more partition identities of the type given in Theorem 1.1. From Jacobi's quartic identity

$$(-q;q^2)_\infty^8 - (q;q^2)_\infty^8 = 16q(-q^2;q^2)_\infty^8 \tag{9}$$

they infer that the number of partitions of $2n+1$ into distinct odd parts is eight times the number of partitions of $2n$ into distinct even parts, provided each integer occurs in eight different copies. Similarly, from

$$(-q;q^2)_\infty^2(-q^3;q^6)_\infty^2 - (q;q^2)_\infty^2(q^3;q^6)_\infty^2 = 4q(-q^2;q^2)_\infty^2(-q^6;q^6)_\infty^2 \tag{10}$$

they infer that the number of partitions of $2n+1$ into distinct odd parts is two times the number of partitions of $2n$ into distinct even parts, provided that multiples of 3 occur in four copies and the remaining integers occur in two copies.

Just like (1), the identities (9) and (10) are easily seen to be specializations of the key-identity (3). Taking $a = 1$ and then making the substitution $(c, bc) \to (a, b)$ yields

$$(11)\quad (-a,-b,-q/a,-q/b;q)_\infty^2-(a,b,q/a,q/b;q)_\infty^2 \\ =4a(-ab,-b/a,-q/ab,-aq/b;q)_\infty(-q;q)_\infty^4.$$

Replacing $q \to q^6$ and then letting $(a, b) \to (q, q^3)$ gives (10), whereas setting $b = a$ gives

$$(12)\quad (-a,-q/a;q)_\infty^4-(a,q/a;q)_\infty^4=8a(-a^2,-q/a^2;q)_\infty(-q;q)_\infty^6.$$

Replacing $q \to q^2$ and then letting $a \to q$ we obtain (9).

Although the two partition theorems given above follow from (3) we should perhaps remark that they are not special cases of Theorem 1.3.

By making different specializations in (11) and (12) it is again possible to generalize the above-stated partition identities to other moduli. For example from (12) with $q \to q^4$ followed by $a \to q$ one finds

$$(-q;q^2)_\infty^4-(q;q^2)_\infty^4=8q(-q^2;q^2)_\infty^2(-q^4;q^4)_\infty^4.$$

This implies that the number of partitions of $2n + 1$ into distinct odd parts is four times the number of partitions of $2n$ into distinct even parts, provided the odd integers occur in four copies, integers congruent to 2 (mod 4) occur in two copies and multiples of four occur in six copies.

Another nice example follows from (11) with $q \to q^8$ followed by $(a, b) \to (q, q^3)$. Then

$$(-q;q^2)_\infty^2-(q;q^2)_\infty^2=4q(-q^2;q^2)_\infty(-q^4;q^4)_\infty(-q^8;q^8)_\infty^2.$$

This implies that the number of partitions of $2n + 1$ into distinct odd parts is two times the number of partitions of $2n$ into distinct even parts, provided the odd integers occur in two copies, integers congruent to 2 (mod 4) occur in one copy, integers congruent to 4 (mod 8) occur in two copies and multiples of eight occur in four copies.

We leave it to the reader to formulate the more general partition theorems arising from (11) and (12).

Finally we wish to point out that it is possible to formulate refinements of each of the partition identities stated in this paper. Such refinements could possibly help in finding bijective proofs. In the case of Theorem 1.1 one can assign a weight $\omega(\lambda)$ to admissible partition λ, where $\omega(\lambda) = \prod_{i\geq 1} \omega_{\lambda_i}$ with ω_{λ_i} the weight of the part λ_i. Here

$$\omega_m=\begin{cases} c^{\pm 1} & \text{if } m\equiv \pm 1 \pmod{14} \\ (ac)^{\pm 1} & \text{if } m\equiv \pm 3 \pmod{14} \\ (bc)^{\pm 1} & \text{if } m\equiv \pm 5 \pmod{14} \\ (abc) & \text{if } m\equiv 7 \pmod{14} \\ (abc)^{-1} & \text{if } m\equiv \overline{7} \pmod{14} \end{cases}$$

for odd parts m and

$$\omega_m = \begin{cases} a^{\pm 1} & \text{if } m \equiv \pm 2 \pmod{14} \\ b^{\pm 1} & \text{if } m \equiv \pm 4 \pmod{14} \\ (abc^2)^{\pm 1} & \text{if } m \equiv \pm 8 \pmod{14} \\ 1 & \text{if } m \equiv 0, \overline{0} \pmod{14} \end{cases}$$

for even parts m. With this definition of the weight of a partition, Theorem 1.1 can be refined to the statement that $\mathcal{E}(n;\omega) = \mathcal{O}(n;c\omega)$ with $\mathcal{E}(n;\omega)$ and $\mathcal{O}(n;\omega)$ defined as before but with the added condition that the partitions being counted should have a fixed weight ω. Returning to the partitions of 18 and 19 we may now add their weights as follows

$$\begin{aligned}
&\omega = b: && (18), (14,4), (\overline{14},4), (12,4,2), (8,6,4) \\
&\omega = a^2: && (16,2) \\
&\omega = (a^2bc^2)^{-1}: && (12,6) \\
&\omega = ac^2: && (10,8) \\
&\omega = (bc)^{-2}: && (10,6,2)
\end{aligned}$$

and

$$\begin{aligned}
&\omega = bc: && (19), (13,5,1), (11,7,1), (11,5,3), (7,\overline{7},5) \\
&\omega = a^2c: && (9,7,3) \\
&\omega = (a^2bc)^{-1}: && (11,\overline{7},1) \\
&\omega = ac^3: && (15,3,1) \\
&\omega = (b^2c)^{-1}: && (9,\overline{7},3).
\end{aligned}$$

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