

A Higher-Level Bailey Lemma: Proof and Application

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Abstract

In a recent letter, new representations were proposed for the pair of sequences (γ, δ) , as defined formally by Bailey in his famous lemma. Here we extend and prove this result, providing pairs (γ, δ) labelled by the Lie algebra A_{N-1} , two non-negative integers ℓ and k and a partition λ , whose parts do not exceed $N - 1$. Our results give rise to what we call a “higher-level” Bailey lemma. As an application it is shown how this lemma can be applied to yield general q -series identities, which generalize some well-known results of Andrews and Bressoud.

1 Introduction

A well-known approach to proving q -series identities of the Rogers–Ramanujan type is the Andrews–Bailey construction [8, 7], given by the following lemma.

Lemma 1.1 *Let $\alpha = \{\alpha_L\}_{L \geq 0}$ and $\beta = \{\beta_L\}_{L \geq 0}$ be sequences which satisfy*

$$\beta_L = \sum_{r=0}^L \frac{\alpha_r}{(q)_{L-r}(aq)_{L+r}}, \quad (1.1)$$

for all L . Then the sequences α' and β' defined by

$$\alpha'_L = \frac{(\rho_1)_L(\rho_2)_L(aq/\rho_1\rho_2)^L}{(aq/\rho_1)_L(aq/\rho_2)_L} \alpha_L$$

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$$\beta'_L = \sum_{r=0}^L \frac{(\rho_1)_r (\rho_2)_r (aq/\rho_1\rho_2)^r (aq/\rho_1\rho_2)_{L-r}}{(aq/\rho_1)_L (aq/\rho_2)_L (q)_{L-r}} \beta_r, \quad (1.2)$$

again satisfy equation (1.1).

A pair of sequences (α, β) that satisfies (1.1) is called a Bailey pair relative to a . The iteration of lemma 1.1:

$$(\alpha, \beta) \rightarrow (\alpha', \beta') \rightarrow (\alpha'', \beta'') \rightarrow \dots$$

is called a Bailey chain. The symbol $(a)_n$ in the above lemma is the usual q -shifted factorial,

$$(a; q)_\infty = (a)_\infty = \prod_{n=0}^{\infty} (1 - aq^n) \quad (1.3)$$

and

$$(a; q)_n = (a)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty} \quad n \in \mathbb{Z}. \quad (1.4)$$

Perhaps not so well-known however, is that Bailey in his original paper [8] considered another pair of sequences $\gamma = \{\gamma_L\}_{L \geq 0}$ and $\delta = \{\delta_L\}_{L \geq 0}$ related by

$$\gamma_L = \sum_{r=L}^{\infty} \frac{\delta_r}{(q)_{r-L} (aq)_{r+L}}, \quad (1.5)$$

for all L . We will refer to a pair of sequences (γ, δ) which satisfies (1.5) as a conjugate Bailey pair relative to a . Bailey noted that given a Bailey pair and some conjugate Bailey pair (both relative to a) the following identity holds:

$$\sum_{L=0}^{\infty} \alpha_L \gamma_L = \sum_{L=0}^{\infty} \beta_L \delta_L. \quad (1.6)$$

As pointed out by Andrews [7], lemma 1.1 is a special case of this result with conjugate Bailey pair

$$\gamma_L = \frac{(\rho_1)_L (\rho_2)_L (aq/\rho_1\rho_2)^L}{(aq/\rho_1)_L (aq/\rho_2)_L} \frac{1}{(q)_{M-L} (aq)_{M+L}} \quad (1.7)$$

and

$$\delta_L = \frac{(\rho_1)_L (\rho_2)_L (aq/\rho_1\rho_2)^L}{(aq/\rho_1)_M (aq/\rho_2)_M} \frac{(aq/\rho_1\rho_2)_{M-L}}{(q)_{M-L}}, \quad (1.8)$$

where M is an arbitrary non-negative integer.

As a special case of (1.7) and (1.8) we may send the undetermined parameters ρ_1 and ρ_2 to infinity. Using

$$\lim_{a \rightarrow \infty} a^{-n} (a)_n = (-1)^n q^{n(n-1)/2}, \quad (1.9)$$

this gives the simple expressions

$$\gamma_L = \frac{a^L q^{L^2}}{(q)_{M-L} (aq)_{M+L}} \quad \text{and} \quad \delta_L = \frac{a^L q^{L^2}}{(q)_{M-L}}. \quad (1.10)$$

It is the aim of this paper to show that these two expressions are the first instances of an infinite series of conjugate Bailey pairs. Specifically, we establish a conjugate Bailey pair relative to $a = q^{k+\ell}$, for each pair of non-negative integers k and ℓ , positive integer N and partition λ with largest part not exceeding $N - 1$. For $k = 0$ and $\lambda = (\ell)$ or $|\lambda| = 0$, $\ell = N$, this result was announced in ref. [36]. The conjugate Bailey pair (1.10) with a of the form $a = q^\ell$ is the special case $N = 1$ and $k = 0$.

Before we proceed we note that Milne and Lilly [30, 32, 33] have recently given higher-dimensional generalizations of the Bailey transform and the Bailey lemma in the setting of very-well poised basic hypergeometric functions on the higher rank groups A_ℓ and C_ℓ . Specifically they generalized the defining relation of a Bailey pair to higher rank (equation (1.1) corresponding to A_1) and obtained the corresponding generalization of lemma 1.1. This has to be contrasted with the results obtained in this paper, which stay within the framework of A_1 , but provide higher-level (spin) representations for conjugate Bailey pairs. Indeed these representations involve expressions closely related to the string functions of the affine Lie algebra $A_1^{(1)}$ at level N , and can be used to obtain q -series identities for the branching functions of the level- N $A_1^{(1)}$ cosets theories.

In the next section we slightly reformulate the definition of a conjugate Bailey pair and state our main result; an infinite hierarchy of conjugate Bailey pairs. In section 3 we present a recursive proof of our result using telescopic expansion techniques. In section 4 we apply what we term the ‘‘higher-level’’ Bailey lemma to derive a very general q -series identity, which contains many known Rogers-Ramanujan type identities as specializations. For the proof of this general identity we also introduce some further transformations of the Bailey chain and Bailey lattice type. We finally conclude with a brief discussion of our results.

2 A hierarchy of conjugate Bailey pairs

In the following we present an infinite series of pairs of sequences (γ, δ) which satisfy equation (1.5). In fact, for notational reasons it turns out to be fruitful to slightly modify the definition (1.5) of a conjugate Bailey pair.

Definition 2.1 *A pair of sequences $\Gamma = \{\Gamma_{L,k}\}_{L,k \geq 0}$ and $\Delta = \{\Delta_{L,k}\}_{L \geq k \geq 0}$ which satisfies*

$$\Gamma_{L,k} = \sum_{r=L+k}^{\infty} \frac{\Delta_{r,k}}{(q)_{r-L-k}(aq)_{r+L}}, \quad (2.1)$$

for all $L, k \geq 0$ is called a (Γ, Δ) -pair relative to a .

At first sight it may seem that a (Γ, Δ) -pair is more general than a conjugate Bailey pair, reducing to it for $k = 0$. However note that given a (Γ, Δ) -pair relative to a and setting

$$\gamma_L^{(k)} = \Gamma_{L,k} \quad \text{and} \quad \delta_L^{(k)} = \frac{\Delta_{L+k,k}}{(aq)_k}, \quad (2.2)$$

the pair $(\gamma^{(k)}, \delta^{(k)})$ becomes a conjugate Bailey pair relative to aq^k .

We now proceed to state a series of (Γ, Δ) -pairs. For this some further notation is needed. First, we need two types of Gaussian polynomials or q -binomial coefficients (see for example [22]).

Definition 2.2 For n, m arbitrary integers

$$\begin{bmatrix} m+n \\ n \end{bmatrix} = \begin{cases} \frac{(q)_{n+m}}{(q)_n(q)_m} & n, m \geq 0 \\ 0 & \text{otherwise,} \end{cases} \quad (2.3)$$

and

$$\begin{bmatrix} m+n \\ n \end{bmatrix}' = \begin{cases} \frac{(q^{n+1})_m}{(q)_m} & m \geq 0 \\ 0 & \text{otherwise.} \end{cases} \quad (2.4)$$

Notice that the primed version of the q -binomials is asymmetric in m and n , and that both types of q -binomials coincide for $m+n \geq 0$. Both the ordinary and the primed binomials enjoy the recurrences

$$\begin{bmatrix} m+n \\ n \end{bmatrix} = \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix} + q^n \begin{bmatrix} m+n-1 \\ n \end{bmatrix} \quad (2.5)$$

$$= \begin{bmatrix} m+n-1 \\ n \end{bmatrix} + q^m \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}, \quad (2.6)$$

with the notable exception that when $n = m = 0$, (2.5) and (2.6) break down for the ordinary q -binomials (2.3).

Next we introduce the following notation. Throughout N is assumed to be a fixed positive integer and $\mathbf{v} = (v_1, \dots, v_{N-1})$ denotes a vector with integer entries, i.e., $\mathbf{v} \in \mathbb{Z}^{N-1}$. In particular, \mathbf{e}_j is the j -th unit vector, $(\mathbf{e}_j)_k = \delta_{j,k}$ ($\delta_{j,k} = 1$ for $j = k$ and 0 otherwise). By definition $\mathbf{e}_0 = \mathbf{e}_N = \mathbf{0}$. For a given matrix M , we use the following notation for matrix multiplication $\mathbf{v}M\mathbf{v} = \sum_{j,k=1}^{N-1} v_j M_{j,k} v_k$, where we omit the transposition symbol. We will often encounter the Cartan matrix of the Lie algebra A_{N-1} , denoted by C . That is, $C_{j,k} = 2\delta_{j,k} - \delta_{j,k-1} - \delta_{j,k+1}$. Further we adopt the compact notation

$$\begin{bmatrix} k \\ \mathbf{v} \end{bmatrix}_q = \frac{(q)_k}{(q)_{v_1} \cdots (q)_{v_{N-1}} (q)_{k-v_1-\cdots-v_{N-1}}} \quad (2.7)$$

for $\mathbf{v} \in \mathbb{Z}_{\geq 0}^{N-1}$ and zero otherwise. Lastly, let $\lambda = (\lambda_1, \lambda_2, \dots)$, $\lambda_1 \geq \lambda_2 \geq \dots$ denote a partition with $\sum_i \lambda_i = |\lambda|$. If λ has largest part $\lambda_1 \leq N-1$, we abbreviate $\mathbf{e}_{\lambda_1} + \mathbf{e}_{\lambda_2} + \dots$ to \mathbf{e}_λ .

We can now state the following theorem.

Theorem 2.1 Fix integers $M \geq 0$, $N \geq 1$ and $\ell \geq 0$, and fix a partition λ whose largest part does not exceed $N-1$. Choose $\sigma \in \{0, 1\}$ such that $\ell + |\lambda| + \sigma N$ is even. Then the following two sequences form a (Γ, Δ) -pair relative to $a = q^\ell$:

$$\begin{aligned} \Gamma_{L,k} &= \frac{a^{L/N+k} q^{L^2/N+kL}}{(q)_{M-L-k} (aq)_{L+M}} \\ &\times \sum_{\boldsymbol{\eta}, \mathbf{i} \in \mathbb{Z}^{N-1}}^{(\sigma)} q^{-\mathbf{i}C^{-1}(\mathbf{i}+(2L+\ell)\mathbf{e}_1)} \begin{bmatrix} k \\ \mathbf{i} \end{bmatrix}_{1/q} q^{\boldsymbol{\eta}C^{-1}(\boldsymbol{\eta}-\mathbf{e}_\lambda)} \prod_{j=1}^{N-1} \begin{bmatrix} \mu_j + \eta_j \\ \eta_j \end{bmatrix}' \end{aligned} \quad (2.8)$$

and

$$\Delta_{L,k} = \frac{a^{L/N} q^{L^2/N - kL}}{(q)_{M-L}} \sum_{\mathbf{n} \in \mathbb{Z}^{N-1}}^{(\sigma)} q^{\mathbf{n} C^{-1}(\mathbf{n} - \mathbf{e}_\lambda)} \prod_{j=1}^{N-1} \begin{bmatrix} m_j + n_j \\ n_j \end{bmatrix}. \quad (2.9)$$

The variables μ_j , which occur in the primed q -binomial in (2.8), are fixed by the (external) variables L, k and the (summation) variables \mathbf{i} and $\boldsymbol{\eta}$ by the equation

$$\boldsymbol{\mu} = C^{-1} \left((M - L - k) \mathbf{e}_1 + (M + L + \ell) \mathbf{e}_{N-1} + \mathbf{e}_\lambda - \sum_{j=1}^{N-1} (i_j + i_{j+1}) \mathbf{e}_j - 2\boldsymbol{\eta} \right), \quad (2.10)$$

where

$$i_N := k - i_1 - \cdots - i_{N-1}. \quad (2.11)$$

Similarly, the variables m_j , which occur in the q -binomial in (2.9), are fixed by L, \mathbf{n} thanks to

$$\mathbf{m} = C^{-1} \left((2L + \ell) \mathbf{e}_{N-1} + \mathbf{e}_\lambda - 2\mathbf{n} \right). \quad (2.12)$$

So far we refrained from defining the superscript (σ) in the sums over $\boldsymbol{\eta}$, \mathbf{i} and \mathbf{n} in (2.8) and (2.9). To explain this notation (as well as the origin of the parity restrictions in the theorem) let us consider expression (2.9). In this expression we sum over the vector \mathbf{n} with integer entries. Given such \mathbf{n} , we can compute a companion vector \mathbf{m} using the (\mathbf{m}, \mathbf{n}) -system (2.12). Of course, because of the product over the q -binomials, we only wish to consider those \mathbf{n} which return a vector \mathbf{m} whose entries are all integer. A close scrutiny of equation (2.12) shows that $\mathbf{m} \in \mathbb{Z}^{N-1}$ if and only if $\ell + |\lambda| + \sigma N$ is even and

$$\frac{L + (\ell - |\lambda|)/2}{N} - (C^{-1}\mathbf{n})_1 \in \mathbb{Z} + \frac{\sigma}{2},$$

where σ takes either the value 0 or 1. A similar analysis can be made for the sum in (2.8) where \mathbf{n} is replaced by $\boldsymbol{\eta} + \mathbf{i}$. More compactly we denote

$$\sum_{\substack{\mathbf{n}_1, \dots, \mathbf{n}_p \in \mathbb{Z}^{N-1} \\ \frac{L + (\ell - |\lambda|)/2}{N} - (C^{-1}(\mathbf{n}_1 + \dots + \mathbf{n}_p))_1 \in \mathbb{Z} + \frac{\sigma}{2}}} = \sum_{\mathbf{n}_1, \dots, \mathbf{n}_p \in \mathbb{Z}^{N-1}}^{(\sigma, L)} = \sum_{\mathbf{n}_1, \dots, \mathbf{n}_p \in \mathbb{Z}^{N-1}}^{(\sigma)}. \quad (2.13)$$

After stating theorem 2.1, it seems appropriate to make a few remarks about its origin. In statistical mechanics there has recently been much interest in so-called fermionic representations of Virasoro characters, see e.g., ref. [19]. Such fermionic character representations relate to combinatorial or “subtractionless” bases for Virasoro modules. One of the most successful approaches to this problem has been the study of exactly solvable lattice models using what is termed the Bethe Ansatz technique. One of the many remarkable features of this technique, is that it gives rise to so-called fermionic polynomials, which in a special limit give fermionic character expressions. (The polynomials featuring in items 287 and 289 of MacMahon’s Combinatory Analysis, Vol. 2 [31], are probably the simplest examples of fermionic polynomials.) The rich combinatorics of the fermionic polynomials are described by systems of (quasi)particles that obey exclusion statistics. These exclusion rules are encoded in what are called (\mathbf{m}, \mathbf{n}) -systems [9]. Here \mathbf{n} is a vector whose j -th entry n_j denotes the occupation number of particles

of type j . The vector \mathbf{m} has an interpretation in terms of occupation numbers of holes or anti-particles.

Returning to the above theorem, equations (2.10) and (2.12) are examples of the aforementioned (\mathbf{m}, \mathbf{n}) -systems¹, and apart from an overall factor, the sequence Δ with $\sigma = 0$ and $\lambda = (\ell)$, coincides with the fermionic polynomials of the level-2 $A_{N-1}^{(1)}$ Jimbo–Miwa–Okado models [27], as obtained in ref. [20].

Maybe more important than theorem 2.1 itself is the following corollary.

Corollary 2.1 *Let M, N, ℓ, λ and σ be as in theorem 2.1. Then the following two sequences form a conjugate Bailey pair relative to $a = q^\ell$:*

$$\gamma_L = \frac{a^{L/N} q^{L^2/N}}{(q)_{M-L} (aq)_{M+L}} \sum_{\boldsymbol{\eta} \in \mathbb{Z}^{N-1}}^{(\sigma)} q^{\boldsymbol{\eta} C^{-1}(\boldsymbol{\eta} - \mathbf{e}_\lambda)} \prod_{j=1}^{N-1} \begin{bmatrix} \mu_j + \eta_j \\ \eta_j \end{bmatrix} \quad (2.14)$$

and

$$\delta_L = \frac{a^{L/N} q^{L^2/N}}{(q)_{M-L}} \sum_{\mathbf{n} \in \mathbb{Z}^{N-1}}^{(\sigma)} q^{\mathbf{n} C^{-1}(\mathbf{n} - \mathbf{e}_\lambda)} \prod_{j=1}^{N-1} \begin{bmatrix} m_j + n_j \\ n_j \end{bmatrix}. \quad (2.15)$$

Here \mathbf{m} is given by (2.12) and $\boldsymbol{\mu}$ by

$$\boldsymbol{\mu} = C^{-1} \left((M-L)\mathbf{e}_1 + (M+L+\ell)\mathbf{e}_{N-1} + \mathbf{e}_\lambda - 2\boldsymbol{\eta} \right). \quad (2.16)$$

We remark that this generalizes the conjugate Bailey pair (1.10) (with the restriction $a = q^\ell$, $\ell \in \mathbb{Z}_{\geq 0}$.)

Proof of corollary 2.1

Setting $k = 0$ in equation (2.9) immediately gives $\Delta_{L,0} = \delta_L$. Setting $k = 0$ in equation (2.8) we find that the sum over \mathbf{i} is non-zero for $\mathbf{i} = \mathbf{0}$ only. Hence (2.10) simplifies to (2.16) and $\Gamma_{L,0}$ gives precisely the above expression for γ_L provided one can show that the primed q -binomials in (2.8) can be replaced by the unprimed q -binomials. To see this, notice that both q -binomials are zero unless $\mu_j \geq 0$ for all $j = 1, \dots, N-1$. From this, and (2.16), it follows that with $\mu_0 = \mu_N = 0$,

$$\mu_j + \eta_j = \frac{1}{2} \left(\mu_{j-1} + \mu_{j+1} + (M-L)\delta_{1,j} + (M+L+\ell)\delta_{N-1,j} + \delta_{\lambda_1,j} + \delta_{\lambda_2,j} + \dots \right) \geq 0. \quad (2.17)$$

(We can safely assume that $M \geq L$ since $\Gamma_{L,0}$ (and γ_L) are non-zero for $M \geq L$ only.) But for $\mu_j + \eta_j \geq 0$ the primed and unprimed q -binomials coincide, and so indeed $\Gamma_{L,0} = \gamma_L$. \square

To conclude this section we note that corollary 2.1 will be used in the following for applications rather than theorem 2.1. Almost all of the identities arising from (2.8) and (2.9) can either be obtained from corollary 2.1 or are of such complexity that they seem of little relevance. Still, the parameter k in theorem 2.1 is indispensable to us, as it gives rise to recurrences which are at the heart of our proof of the infinite series of conjugate Bailey pairs.

¹For the ‘‘interpretation’’ of the $(\boldsymbol{\mu}, \boldsymbol{\eta})$ -system (2.10), one in fact has to allow for the occupation numbers η_j to become negative. A phenomenon discussed in [11, 13].

3 Proof of theorem 2.1

In this section we prove theorem 2.1 using recurrence relations. For the proof of these recurrences we rely on the following generalizations of the q -binomial recurrences (2.6), termed telescopic expansions [9]. (Similar generalizations of (2.5) will not be needed here.)

Lemma 3.1 *For integer $N \geq 2$ and A_j, B_j integer for all $j = 1, \dots, N-1$,*

$$\prod_{j=1}^{N-1} \begin{bmatrix} A_j + B_j \\ A_j \end{bmatrix}' = \prod_{j=1}^{N-1} \begin{bmatrix} A_j + B_j - 1 \\ A_j \end{bmatrix}' + \sum_{p=1}^{N-1} q^{B_p} \prod_{j=1}^{N-1} \begin{bmatrix} A_j + B_j - \chi(j \leq p) \\ A_j - \delta_{j,p} \end{bmatrix}' \quad (3.1)$$

and

$$\prod_{j=1}^{N-1} \begin{bmatrix} A_j + B_j \\ A_j \end{bmatrix}' = \prod_{j=1}^{N-1} \begin{bmatrix} A_j + B_j - 1 \\ A_j \end{bmatrix}' + \sum_{p=1}^{N-1} q^{B_p} \prod_{j=1}^{N-1} \begin{bmatrix} A_j + B_j - \chi(j \geq p) \\ A_j - \delta_{j,p} \end{bmatrix}' \quad (3.2)$$

where $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$.

Proof. For $N = 2$ (3.1) and (3.2) simplify to (2.6). For $N \geq 2$, (3.1) and (3.2) follow by induction on N and application of (2.6). \square

Let us now return to theorem 2.1. Inserting both the definition of Δ and Γ into equation (2.1) and reshuffling some of the terms, the claim of theorem 2.1 can be re-expressed as the following polynomial identity.

Lemma 3.2 *Let M, N, ℓ, λ and σ be as in theorem 2.1. Then for $0 \leq L \leq M$ and $0 \leq k \leq M - L$,*

$$\begin{aligned} & \sum_{\boldsymbol{\eta}, \mathbf{i} \in \mathbb{Z}^{N-1}} q^{-\mathbf{i} C^{-1}(\mathbf{i} + (2L + \ell)\mathbf{e}_1)} \begin{bmatrix} k \\ \mathbf{i} \end{bmatrix}_{1/q} q^{\boldsymbol{\eta} C^{-1}(\boldsymbol{\eta} - \mathbf{e}_\lambda)} \prod_{j=1}^{N-1} \begin{bmatrix} \eta_j + \mu_j \\ \eta_j \end{bmatrix}' \\ &= \sum_{r=L+k}^M q^{(r+L+\ell)(r-L-Nk)/N} \begin{bmatrix} M-L-k \\ M-r \end{bmatrix} \frac{(q^{\ell+1})_{L+M}}{(q^{\ell+1})_{L+r}} \\ & \quad \times \sum_{\mathbf{n} \in \mathbb{Z}^{N-1}} q^{\mathbf{n} C^{-1}(\mathbf{n} - \mathbf{e}_\lambda)} \prod_{j=1}^{N-1} \begin{bmatrix} m_j + n_j \\ n_j \end{bmatrix}, \end{aligned} \quad (3.3)$$

with $(\boldsymbol{\mu}, \boldsymbol{\eta})$ -system (2.10) and (\mathbf{m}, \mathbf{n}) -system (2.12), where L in (2.12) has been replaced by r .

3.1 Proof of lemma 3.2

We prove the polynomial identity (3.3) by induction on k and $M - L$. To this end we show that both the left- and right-hand side of (3.3), denoted by f_1 and f_2 , respectively, obey the recursion relations

$$f(M, L, k) = f(M-1, L, k) + q^{M+L+\ell} \left(f(M, L, k+1) - f(M-1, L, k) \right) \quad \text{for } 0 \leq k < M-L \quad (3.4)$$

$$f(M, L, k) = q^{-(2L+\ell+1)(N-1)/N} f(M, L+1, k-1) \quad \text{for } k = M-L \quad (3.5)$$

Figure 1: Graphical representation of the “flow” induced by the recurrence relations (3.4)–(3.6) in the $(k, M - L)$ -plane.

as well as the initial condition

$$f_1(M, M, 0) = f_2(M, M, 0) \quad \text{for } M \geq 0. \quad (3.6)$$

Equations (3.4)–(3.6) fix the function f uniquely. With (3.6) as a seed, recurrence (3.5) gives $f(M, L, k)$, for all $k = M - L$. Now assume that $f(M, L, k)$ is known for all $k = M - L - n$, with $n = 0, \dots, n_0$. Since (3.4) expresses each $f(M, L, k)$ with $k = M - L - n - 1$ as the sum of two $f(M, L, k)$ with $k = M - L - n$, this implies that $f(M, L, k)$ is fixed for $k = M - L - (n_0 + 1)$. Since the assumption is obviously true for $n_0 = 0$, we indeed find that f is fixed by (3.4)–(3.6).

The recurrences are illustrated in figure 1. Each coordinate in the $(k, M - L)$ -plane for which $0 \leq L \leq M$ and $0 \leq k \leq M - L$ is represented by a node, the node at $(k, M - L)$ representing $f(M, L, k)$. All incoming arrows at a given node indicate which values of k and $M - L$ are needed in the recursion relations. For example the nodes on the diagonal $k = M - L$ all follow from just the previous node on the diagonal as given by (3.5). All other points need two values of k and $M - L$ as given by (3.4).

Proof of the initial condition (3.6).

For $k = 0$ and $L = M$, f_1 and f_2 reduce to

$$f_1(M, M, 0) = \sum_{\boldsymbol{\eta} \in \mathbb{Z}^{N-1}}^{(\sigma, M)} q^{\boldsymbol{\eta} C^{-1}(\boldsymbol{\eta} - \mathbf{e}_\lambda)} \prod_{j=1}^{N-1} \left[\begin{matrix} \mu_j + \eta_j \\ \eta_j \end{matrix} \right]', \quad (3.7)$$

with $\boldsymbol{\mu} = C^{-1}((2M + \ell)\mathbf{e}_{N-1} + \mathbf{e}_\lambda - 2\boldsymbol{\eta})$, and

$$f_2(M, M, 0) = \sum_{\mathbf{n} \in \mathbb{Z}^{N-1}}^{(\sigma, M)} q^{\mathbf{n} C^{-1}(\mathbf{n} - \mathbf{e}_\lambda)} \prod_{j=1}^{N-1} \left[\begin{matrix} m_j + n_j \\ n_j \end{matrix} \right], \quad (3.8)$$

with $\mathbf{m} = C^{-1}((2M + \ell)\mathbf{e}_{N-1} + \mathbf{e}_\lambda - 2\mathbf{n})$. Since $\eta_j + \mu_j \geq 0$, we can remove the prime in (3.7) and we are done. \square

Since the (\mathbf{m}, \mathbf{n}) -system (2.12) corresponding to the right-hand side of (3.3) does not depend on the variables $M - L$ and k , f_2 is much easier to handle than f_1 . Hence we proceed in reversed order.

Proof of the recursion relation (3.4) for f_2

We substitute the definition of f_2 , given by the right-hand side of (3.3), into $f_2(M, L, k) - (1 - q^{M+L+\ell})f_2(M-1, L, k)$. This immediately yields the desired result since

$$\begin{aligned}
& f_2(M, L, k) - (1 - q^{M+L+\ell})f_2(M-1, L, k) \\
&= \sum_{r=L+k}^M q^{(r+L+\ell)(r-L-Nk)/N} \left(\begin{bmatrix} M-L-k \\ M-r \end{bmatrix} - \begin{bmatrix} M-L-k-1 \\ M-r-1 \end{bmatrix} \right) \\
&\quad \times \frac{(q^{\ell+1})_{L+M}}{(q^{\ell+1})_{L+r}} \sum_{\mathbf{n} \in \mathbb{Z}^{N-1}}^{(\sigma, r)} q^{\mathbf{n}C^{-1}(\mathbf{n}-\mathbf{e}_\lambda)} \prod_{j=1}^{N-1} \begin{bmatrix} m_j + n_j \\ n_j \end{bmatrix} \\
&= q^{M+L+\ell} f_2(M, L, k+1), \tag{3.9}
\end{aligned}$$

where in the last step we have used equation (2.5). \square

Proof of the recursion relation (3.5) for f_2

The proof of (3.5) for f_2 is again trivial, since

$$\begin{aligned}
& q^{-(2L+\ell+1)(N-1)/N} f_2(M, L+1, M-L-1) \\
&= q^{(M+L+\ell)(M-L-N(M-L))/N} \sum_{\mathbf{n} \in \mathbb{Z}^{N-1}}^{(\sigma, M)} q^{\mathbf{n}C^{-1}(\mathbf{n}-\mathbf{e}_\lambda)} \prod_{j=1}^{N-1} \begin{bmatrix} m_j + n_j \\ n_j \end{bmatrix} \\
&= f_2(M, L, M-L). \tag{3.10}
\end{aligned}$$

\square

So far, everything has been extremely simple. The proof that f_1 satisfies the recurrences (3.4) and (3.5) is however considerably more involved.

Proof of the recursion relations (3.4) for f_1

To avoid possible confusion in the course of the proof, we will make the dependence on M, L, k and \mathbf{i} of the vector $\boldsymbol{\mu}$ explicit by writing $\boldsymbol{\mu}(M, L, k, \mathbf{i})$. Most importantly, the left-hand side of (2.10) should now be read as $\boldsymbol{\mu}(M, L, k, \mathbf{i})$. We also make repeated use of the matrix elements of the inverse Cartan matrix, given by

$$C_{j,k}^{-1} = \begin{cases} (N-k)j/N & \text{for } j \leq k \\ C_{k,j}^{-1} & \text{for } j > k. \end{cases} \tag{3.11}$$

From the definition of f_1 as the left-hand side of (3.3) we have

$$\begin{aligned}
& f_1(M, L, k) - f_1(M-1, L, k) \\
&= \sum_{\boldsymbol{\eta}, \mathbf{i} \in \mathbb{Z}^{N-1}}^{(\sigma, L)} q^{-\mathbf{i}C^{-1}(\mathbf{i}+(2L+\ell)\mathbf{e}_1)+\boldsymbol{\eta}C^{-1}(\boldsymbol{\eta}-\mathbf{e}_\lambda)} \begin{bmatrix} k \\ \mathbf{i} \end{bmatrix}_{1/q} \\
&\quad \times \left\{ \prod_{j=1}^{N-1} \begin{bmatrix} \eta_j + \mu_j(M, L, k, \mathbf{i}) \\ \eta_j \end{bmatrix}' - \prod_{j=1}^{N-1} \begin{bmatrix} \eta_j + \mu_j(M, L, k, \mathbf{i}) - 1 \\ \eta_j \end{bmatrix}' \right\}. \tag{3.12}
\end{aligned}$$

Here we have used $\mu_j(M-1, L, k, \mathbf{i}) = \mu_j(M, L, k, \mathbf{i}) - C_{1,j}^{-1} - C_{N-1,j}^{-1} = \mu_j(M, L, k, \mathbf{i}) - 1$.

We now insert the telescopic expansion (3.1) with the replacements $A_j \rightarrow \eta_j$ and $B_j \rightarrow \mu_j$,

and change $\eta_p \rightarrow \eta_p + 1$ and $i_p \rightarrow i_p - 1$ in the p -th term of the sum. Carrying out both variable changes, using (2.10) and definition (2.11) of i_N , yields

$$\begin{aligned} \mu_j(M, L, k, \mathbf{i}) &\rightarrow \mu_j(M, L, k, \mathbf{i}) + \mathbf{e}_j C^{-1}(\mathbf{e}_{p-1} - \mathbf{e}_p - \mathbf{e}_{N-1}) \\ &= \mu_j(M, L, k, \mathbf{i}) - \chi(j \geq p). \end{aligned} \quad (3.13)$$

As a result (3.12) becomes (note that $\sum_{\boldsymbol{\eta} + \mathbf{e}_p, \mathbf{i} - \mathbf{e}_p}^{(\sigma)} \cdots = \sum_{\boldsymbol{\eta}, \mathbf{i}}^{(\sigma)} \cdots$)

$$\begin{aligned} &f_1(M, L, k) - f_1(M - 1, L, k) \\ &= \sum_{p=1}^{N-1} \sum_{\boldsymbol{\eta}, \mathbf{i} \in \mathbb{Z}^{N-1}}^{(\sigma, L)} q^{-\mathbf{i} C^{-1}(\mathbf{i} + (2L + \ell)\mathbf{e}_1) + \boldsymbol{\eta} C^{-1}(\boldsymbol{\eta} - \mathbf{e}_\lambda)} \left[\begin{matrix} k \\ \mathbf{i} - \mathbf{e}_p \end{matrix} \right]_{1/q} \\ &\quad \times q^{\mu_p(M, L, k, \mathbf{i}) - 1 + \mathbf{e}_p C^{-1}(2\boldsymbol{\eta} + 2\mathbf{i} + (2L + \ell)\mathbf{e}_1 - \mathbf{e}_\lambda)} \prod_{j=1}^{N-1} \left[\begin{matrix} \eta_j + \mu_j(M, L, k, \mathbf{i}) - 1 \\ \eta_j \end{matrix} \right]'. \end{aligned} \quad (3.14)$$

Using (2.10) and (3.11) one may readily check that

$$\mu_p(M, L, k, \mathbf{i}) + \mathbf{e}_p C^{-1}(2\boldsymbol{\eta} + 2\mathbf{i} + (2L + \ell)\mathbf{e}_1 - \mathbf{e}_\lambda) = M + L + \ell - k + \sum_{j=1}^p i_j, \quad (3.15)$$

resulting in

$$\begin{aligned} &f_1(M, L, k) - f_1(M - 1, L, k) \\ &= q^{M+L+\ell} \sum_{\boldsymbol{\eta}, \mathbf{i} \in \mathbb{Z}^{N-1}}^{(\sigma, L)} q^{-\mathbf{i} C^{-1}(\mathbf{i} + (2L + \ell)\mathbf{e}_1)} \left(\sum_{p=1}^{N-1} q^{-k-1 + \sum_{j=1}^p i_j} \left[\begin{matrix} k \\ \mathbf{i} - \mathbf{e}_p \end{matrix} \right]_{1/q} \right) \\ &\quad \times q^{\boldsymbol{\eta} C^{-1}(\boldsymbol{\eta} - \mathbf{e}_\lambda)} \prod_{j=1}^{N-1} \left[\begin{matrix} \eta_j + \mu_j(M, L, k, \mathbf{i}) - 1 \\ \eta_j \end{matrix} \right]'. \end{aligned} \quad (3.16)$$

One can now simplify the sum over p . Using definition (2.11) we get

$$\begin{aligned} \sum_{p=1}^{N-1} q^{k+1 - \sum_{j=1}^p i_j} \left[\begin{matrix} k \\ \mathbf{i} - \mathbf{e}_p \end{matrix} \right]_q &= \frac{q^{k+1} (q)_k}{(q)_{i_1} \cdots (q)_{i_{N-1}} (q)_{i_{N+1}}} \sum_{p=1}^{N-1} (1 - q^{i_p}) q^{-\sum_{j=1}^p i_j} \\ &= \frac{(q)_k}{(q)_{i_1} \cdots (q)_{i_{N-1}} (q)_{i_{N+1}}} \left((1 - q^{k+1}) - (1 - q^{i_{N+1}}) \right) \\ &= \left[\begin{matrix} k+1 \\ \mathbf{i} \end{matrix} \right]_q - \left[\begin{matrix} k \\ \mathbf{i} \end{matrix} \right]_q. \end{aligned} \quad (3.17)$$

Substituting this into (3.16) and noting $\mu(M, L, k, \mathbf{i}) - 1 = \mu(M, L, k + 1, \mathbf{i}) = \mu(M - 1, L, k, \mathbf{i})$, we arrive at the desired result

$$f_1(M, L, k) - f_1(M - 1, L, k) = q^{M+L+\ell} \left(f_1(M, L, k + 1) - f_1(M - 1, L, k) \right). \quad (3.18)$$

□

Proof of recursion relation (3.5) for f_1

From the definition of f_1 as the left-hand side of (3.3) we obtain

$$f_1(M, L+1, M-L-1) = \sum_{\boldsymbol{\eta}, \mathbf{i} \in \mathbb{Z}^{N-1}}^{(\sigma, L+1)} q^{-\mathbf{i}C^{-1}(\mathbf{i}+(2L+\ell+2)\mathbf{e}_1)} \left[\begin{matrix} M-L-1 \\ \mathbf{i} \end{matrix} \right]_{1/q} \\ \times q^{\boldsymbol{\eta}C^{-1}(\boldsymbol{\eta}-\mathbf{e}_\lambda)} \prod_{j=1}^{N-1} \left[\begin{matrix} \eta_j + \mu_j(M, L+1, M-L-1, \mathbf{i}) \\ \eta_j \end{matrix} \right]'. \quad (3.19)$$

We now apply telescopic expansion (3.2), with $A_j \rightarrow \eta_j$ and $B_j \rightarrow \mu_j$, to rewrite the product over the q -binomials. Then we make the change of variables $i_1 \rightarrow i_1 - 1$ in the first-term on the right-hand side of (3.2), $\eta_p \rightarrow \eta_p + 1$, $i_{p+1} \rightarrow i_{p+1} - 1$ in the p -th term of the sum ($1 \leq p \leq N-2$) and $\eta_{N-1} \rightarrow \eta_{N-1} + 1$ in the term with $p = N-1$. Since under these variable changes

$$\mu_j(M, L+1, M-L-1, \mathbf{i}) \rightarrow \mu_j(M, L, M-L, \mathbf{i}) - \mathbf{e}_j C^{-1}(\mathbf{e}_p - \mathbf{e}_{p+1} - \mathbf{e}_{N-1}) \\ = \mu_j(M, L, M-L, \mathbf{i}) + \chi(j > p), \quad (3.20)$$

we obtain

$$f_1(M, L+1, M-L-1) \\ = \sum_{p=0}^{N-1} \sum_{\boldsymbol{\eta}, \mathbf{i} \in \mathbb{Z}^{N-1}}^{(\sigma, L)} q^{-\mathbf{i}C^{-1}(\mathbf{i}+(2L+\ell)\mathbf{e}_1) + \boldsymbol{\eta}C^{-1}(\boldsymbol{\eta}-\mathbf{e}_\lambda)} \left[\begin{matrix} M-L-1 \\ \mathbf{i} - \mathbf{e}_{p+1} \end{matrix} \right]_{1/q} \\ \times q^{\mu_p(M, L, M-L, \mathbf{i}) - 2\mathbf{i}C^{-1}(\mathbf{e}_1 - \mathbf{e}_{p+1}) + \mathbf{e}_p C^{-1}(2\boldsymbol{\eta} - \mathbf{e}_\lambda + \mathbf{e}_p) + \mathbf{e}_{p+1} C^{-1}((2L+\ell+2)\mathbf{e}_1 - \mathbf{e}_{p+1})} \\ \times \prod_{j=1}^{N-1} \left[\begin{matrix} \eta_j + \mu_j(M, L, M-L, \mathbf{i}) \\ \eta_j \end{matrix} \right]'. \quad (3.21)$$

Now use (2.10) as well as the definition of i_N (recall that $k = M-L$) to observe that the exponent of q in the third line of the above equation simplifies to

$$(2L+\ell+1)(N-1)/N - \sum_{j=1}^p i_j. \quad (3.22)$$

This allows (3.21) to be reduced to

$$q^{-(2L+\ell+1)(N-1)/N} f_1(M, L+1, M-L-1) \\ = \sum_{\boldsymbol{\eta}, \mathbf{i} \in \mathbb{Z}^{N-1}}^{(\sigma, L)} q^{-\mathbf{i}C^{-1}(\mathbf{i}+(2L+\ell)\mathbf{e}_1)} \left(\sum_{p=0}^{N-1} q^{-\sum_{j=1}^p i_j} \left[\begin{matrix} M-L-1 \\ \mathbf{i} - \mathbf{e}_{p+1} \end{matrix} \right]_{1/q} \right) \\ \times q^{\boldsymbol{\eta}C^{-1}(\boldsymbol{\eta}-\mathbf{e}_\lambda)} \left[\begin{matrix} \eta_j + \mu_j(M, L, M-L, \mathbf{i}) \\ \eta_j \end{matrix} \right]'. \quad (3.23)$$

Again we may simplify the sum over p ,

$$\sum_{p=0}^{N-1} q^{\sum_{j=1}^p i_j} \left[\begin{matrix} M-L-1 \\ \mathbf{i} - \mathbf{e}_{p+1} \end{matrix} \right]_q = \frac{(q)_{M-L-1}}{(q)_{i_1} \cdots (q)_{i_N}} \sum_{p=0}^{N-1} (1 - q^{i_{p+1}}) q^{\sum_{j=1}^p i_j} \\ = \left[\begin{matrix} M-L \\ \mathbf{i} \end{matrix} \right]_q, \quad (3.24)$$

so that the right-hand side of (3.23) gives $f_1(M, L, M-L)$ as desired. \square

4 Applications

We now show how theorem 2.1, and in particular corollary 2.1, can be exploited to obtain very general q -series identities, which take many of the known Rogers–Ramanujan type identities as special cases.

We substitute the conjugate Bailey pair of corollary 2.1 into equation 1.6 and take the limit $M \rightarrow \infty$. This gives the important result

Corollary 4.1 (higher-level Bailey lemma) *Let N, ℓ, λ and σ be as in theorem 2.1, and let (α, β) form a Bailey pair relative to $a = q^\ell$. Then the following identity holds:*

$$\begin{aligned} & \frac{1}{(aq)_\infty} \sum_{L=0}^{\infty} a^{L/N} q^{L^2/N} \alpha_L \sum_{\boldsymbol{\eta} \in \mathbb{Z}^{N-1}}^{(\sigma, L)} \frac{q^{\boldsymbol{\eta} C^{-1}(\boldsymbol{\eta} - \mathbf{e}_\lambda)}}{(q)_{\eta_1} \cdots (q)_{\eta_{N-1}}} \\ &= \sum_{L=0}^{\infty} a^{L/N} q^{L^2/N} \beta_L \sum_{\mathbf{n} \in \mathbb{Z}^{N-1}}^{(\sigma, L)} q^{\mathbf{n} C^{-1}(\mathbf{n} - \mathbf{e}_\lambda)} \prod_{j=1}^{N-1} \begin{bmatrix} m_j + n_j \\ n_j \end{bmatrix}, \end{aligned} \quad (4.1)$$

with the (\mathbf{m}, \mathbf{n}) -system

$$\mathbf{m} = C^{-1} \left((2L + \ell) \mathbf{e}_{N-1} + \mathbf{e}_\lambda - 2\mathbf{n} \right). \quad (4.2)$$

For $\lambda = (\ell)$, $\sigma = 0$ and for $|\lambda| = 0$, $\ell = N$, $\sigma = 1$ this result was announced in ref. [36].

As first noted by Foda and Quano [21] for some specific examples and in full in refs. [10, 13], polynomial identities related to the characters of $c < 1$ Virasoro algebras give rise to Bailey pairs. The name of the above lemma has been motivated by the fact that inserting these Bailey pairs into (4.1) yields Rogers–Ramanujan type identities for the branching functions of level- N $A_1^{(1)}$ conformal coset models. Details of this application will be presented elsewhere [14].² Instead, we apply here corollary 4.1 to some of the simplest known Bailey pairs. For these we wish to use corollary 4.1 as efficiently as possible. To this end we utilize, as well as the Bailey chain and its generalization to the Bailey lattice, two new transformations acting on Bailey pairs.

4.1 Transformations on Bailey pairs

Lemma 1.1 provides a powerful mechanism to obtain new Bailey pairs from a given Bailey pair. One of the limitations of (1.2) is however that if (α, β) is a Bailey pair relative to a , so will be the resulting Bailey pair (α', β') .

A useful transformation which does not keep a fixed was found by Agarwal, Andrews and Bressoud [1, 18]

Lemma 4.1 *Let (α, β) be a Bailey pair relative to a and define $\alpha'_0 = \alpha_0$ and*

$$\begin{aligned} \alpha'_L &= \frac{(\rho_1)_L (\rho_2)_L (a/\rho_1 \rho_2)^L}{(a/\rho_1)_L (a/\rho_2)_L} \left(\frac{(a)_{2L}}{(aq)_{2L}} \alpha_L - a q^{2L-2} \frac{(a)_{2L-2}}{(aq)_{2L-2}} \alpha_{L-1} \right) \quad L \geq 1 \\ \beta'_L &= \sum_{r=0}^L \frac{(\rho_1)_r (\rho_2)_r (a/\rho_1 \rho_2)^r (a/\rho_1 \rho_2)_{L-r}}{(a/\rho_1)_L (a/\rho_2)_L (q)_{L-r}} \beta_r. \end{aligned} \quad (4.3)$$

Then (α', β') is a Bailey pair relative to aq^{-1} .

²It is intriguing to note [12] that for $N = 2$ the characters of these coset models (which are obtained from the branching functions by summing over both parities of σ) also arise from the “classical” conjugate Bailey pair (1.7) and (1.8), taking $\rho_1, M \rightarrow \infty$ and setting $\rho_2 = -(aq)^{1/2}$.

In combination with lemma 1.1, transformation (4.3) gives rise to what is called the Bailey lattice.

We now introduce two new transformations that map a Bailey pair onto a new Bailey pair. The first transformation is of the type (1.2) in that it leaves the value of a invariant.

Lemma 4.2 *If (α, β) forms a Bailey pair relative to a , then the sequences*

$$\begin{aligned}\alpha'_L &= a^L q^{L(L+1)} \sum_{r=0}^L \frac{(\rho_1)_r (\rho_2)_r (q^{1-r} / \rho_1 \rho_2)^r}{(aq/\rho_1)_r (aq/\rho_2)_r} \alpha_r \\ &\quad - a^{L-1} q^{L(L-1)} \sum_{r=0}^{L-1} \frac{(\rho_1)_r (\rho_2)_r (q^{1-r} / \rho_1 \rho_2)^r}{(aq/\rho_1)_r (aq/\rho_2)_r} \alpha_r \\ \beta'_L &= q^L \sum_{r=0}^L \frac{(\rho_1)_r (\rho_2)_r (aq/\rho_1 \rho_2)^r (aq/\rho_1 \rho_2)^{L-r}}{(aq/\rho_1)_L (aq/\rho_2)_L (q)^{L-r}} \beta_r.\end{aligned}\tag{4.4}$$

again form a Bailey pair relative to a .

The second transformation closely mimics lemma 4.1.

Lemma 4.3 *Let (α, β) be a Bailey pair relative to a and define $\alpha'_0 = \alpha_0$ and*

$$\begin{aligned}\alpha'_L &= \frac{(\rho_1)_L (\rho_2)_L (aq/\rho_1 \rho_2)^L (a)_{2L}}{(aq/\rho_1)_L (aq/\rho_2)_L (aq)_{2L}} \alpha_L \\ &\quad - \frac{(\rho_1)_{L-1} (\rho_2)_{L-1} (aq/\rho_1 \rho_2)^{L-1}}{(aq/\rho_1)_{L-1} (aq/\rho_2)_{L-1}} aq^{2L-2} \frac{(a)_{2L-2}}{(aq)_{2L-2}} \alpha_{L-1} \quad L \geq 1 \\ \beta'_L &= \sum_{r=0}^L \frac{(\rho_1)_r (\rho_2)_r (aq/\rho_1 \rho_2)^r (aq/\rho_1 \rho_2)^{L-r}}{(aq/\rho_1)_L (aq/\rho_2)_L (q)^{L-r}} \beta_r.\end{aligned}\tag{4.5}$$

Then (α', β') is a Bailey pair relative to aq^{-1} .

Proof of the lemmas 4.2 and 4.3

For both the proof of equation (4.4) and (4.5) we only need the q -analogue of Saalschütz's theorem (equation (3.3.12) of ref. [6]). First, to prove (4.4),

$$\begin{aligned}\beta'_L &= q^L \sum_{k=0}^L \frac{(\rho_1)_k (\rho_2)_k (aq/\rho_1 \rho_2)^k (aq/\rho_1 \rho_2)^{L-k}}{(aq/\rho_1)_L (aq/\rho_2)_L (q)^{L-k}} \beta_k \\ &= q^L \sum_{k=0}^L \frac{(\rho_1)_k (\rho_2)_k (aq/\rho_1 \rho_2)^k (aq/\rho_1 \rho_2)^{L-k}}{(aq/\rho_1)_L (aq/\rho_2)_L (q)^{L-k}} \sum_{m=0}^k \frac{\alpha_m}{(q)_{k-m} (aq)_{k+m}} \\ &= q^L \sum_{m=0}^L \alpha_m \sum_{k=m}^L \frac{(\rho_1)_k (\rho_2)_k (aq/\rho_1 \rho_2)^k (aq/\rho_1 \rho_2)^{L-k}}{(aq/\rho_1)_L (aq/\rho_2)_L (q)^{L-k} (q)_{k-m} (aq)_{k+m}} \\ &= q^L \sum_{m=0}^L \frac{(\rho_1)_m (\rho_2)_m (aq/\rho_1 \rho_2)^m \alpha_m}{(aq/\rho_1)_m (aq/\rho_2)_m (q)^{L-m} (aq)_{L+m}} \\ &= \sum_{m=0}^L \frac{(\rho_1)_m (\rho_2)_m (q^{1-m} / \rho_1 \rho_2)^m \alpha_m}{(aq/\rho_1)_m (aq/\rho_2)_m} \left(\sum_{k=m}^L aq^{2k} - \sum_{k=m+1}^L \right) \frac{a^{k-1} q^{k(k-1)}}{(q)_{L-k} (aq)_{L+k}}\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^L \frac{a^k q^{k(k+1)}}{(q)_{L-k} (aq)_{L+k}} \sum_{m=0}^k \frac{(\rho_1)_m (\rho_2)_m (q^{1-m}/\rho_1 \rho_2)^m \alpha_m}{(aq/\rho_1)_m (aq/\rho_2)_m} \\
&\quad - \sum_{k=0}^L \frac{a^{k-1} q^{k(k-1)}}{(q)_{L-k} (aq)_{L+k}} \sum_{m=0}^{k-1} \frac{(\rho_1)_m (\rho_2)_m (q^{1-m}/\rho_1 \rho_2)^m \alpha_m}{(aq/\rho_1)_m (aq/\rho_2)_m} \\
&= \sum_{k=0}^L \frac{\alpha'_k}{(q)_{L-k} (aq)_{L+k}}. \tag{4.6}
\end{aligned}$$

To prove (4.5) we proceed as follows,

$$\begin{aligned}
\beta'_L &= \sum_{k=0}^L \frac{(\rho_1)_k (\rho_2)_k (aq/\rho_1 \rho_2)^k (aq/\rho_1 \rho_2)_{L-k}}{(aq/\rho_1)_L (aq/\rho_2)_L (q)_{L-k}} \beta_k \\
&= \sum_{m=0}^L \frac{(\rho_1)_m (\rho_2)_m (aq/\rho_1 \rho_2)^m \alpha_m}{(aq/\rho_1)_m (aq/\rho_2)_m (q)_{L-m} (aq)_{L+m}} \\
&= \sum_{m=0}^L \frac{(\rho_1)_m (\rho_2)_m (aq/\rho_1 \rho_2)^m (a)_{2m} \alpha_m}{(aq/\rho_1)_m (aq/\rho_2)_m (aq)_{2m}} \left(\frac{1}{(q)_{L-m} (a)_{L+m}} - \frac{aq^{2m}}{(q)_{L-m-1} (a)_{L+m+1}} \right) \\
&= \sum_{m=0}^L \frac{\alpha'_m}{(q)_{L-m} (a)_{L+m}}. \tag{4.7}
\end{aligned}$$

□

4.2 A general q -series identity

Equipped with corollary 4.1 and the lemmas 1.1, 4.1–4.3, we are prepared to prove the following q -identity.

Theorem 4.1 *Fix integers $N \geq 1$, $\delta \in \{0, 1\}$, $k \geq 2$ and $1 \leq i \leq k$. Let λ be a partition with largest part $\leq N - 1$ and $\sigma \in \{0, 1\}$ such that $|\lambda| + \sigma N$ even. Then the following identity holds:*

$$\begin{aligned}
&\frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} (-1)^j q^{((2k+\delta-2+2/N)j+2k-2i+\delta)j/2} \sum_{\boldsymbol{\eta} \in \mathbb{Z}^{N-1}}^{(\sigma, j)} \frac{q^{\boldsymbol{\eta} C^{-1}(\boldsymbol{\eta} - \mathbf{e}_\lambda)}}{(q)_{\eta_1} \cdots (q)_{\eta_{N-1}}} \\
&= \sum_{r_1 \geq \dots \geq r_{k-1} \geq 0} \frac{q^{r_1^2/N + r_2^2 + \dots + r_{k-1}^2 + r_i + \dots + r_{k-1}}}{(q)_{r_1 - r_2} \cdots (q)_{r_{k-2} - r_{k-1}} (q^{2-\delta}; q^{2-\delta})_{r_{k-1}}} \\
&\quad \times \sum_{\mathbf{n} \in \mathbb{Z}^{N-1}}^{(\sigma, r_1)} q^{\mathbf{n} C^{-1}(\mathbf{n} - \mathbf{e}_\lambda)} \prod_{j=1}^{N-1} \begin{bmatrix} m_j + n_j \\ n_j \end{bmatrix} \tag{4.8}
\end{aligned}$$

with (\mathbf{m}, \mathbf{n}) -system $\mathbf{m} = C^{-1}(2r_1 \mathbf{e}_{N-1} + \mathbf{e}_\lambda - 2\mathbf{n})$.

(Note: the summation symbol $\sum^{(\sigma, a)}$ in the above is that defined in (2.13) with $\ell = 0$.)

Before we prove this result, we note that identity (4.8) has a number of important specializations. First, setting $N = 1$ and rewriting the left-hand side using Jacobi's triple product identity (equation (2.2.10) of ref. [6]), we get

Corollary 4.2 For integers $\delta = 0, 1$, $k \geq 2$ and $1 \leq i \leq k + \delta - 1$,

$$\sum_{n_1, \dots, n_{k-1} \geq 0} \frac{q^{N_1^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1}}}{(q)_{n_1} \dots (q)_{n_{k-2}} (q^{2-\delta}; q^{2-\delta})_{n_{k-1}}} = \prod_{\substack{j=1 \\ j \not\equiv 0, \pm i \pmod{2k+\delta}}}^{\infty} (1 - q^j)^{-1}, \quad (4.9)$$

where $N_j = n_j + \dots + n_{k-1}$.

For $\delta = 1$, $k = 2$ these are the Rogers–Ramanujan identities [34, 35]. For $\delta = 1$, general k , these are Andrews’ analytic counterpart of Gordon’s partition theorem [4, 24]. For $\delta = 0$ the identities (4.9) were discovered by Bressoud [16], and are the analytic counterpart of partition identities of Andrews [2] and Bressoud [15]. We note that the derivation of these identities using the Bailey chain and lattice is of course not new, see e.g., ref. [1].

The next two corollaries are obtained by setting $N = 2$ in theorem 4.1

Corollary 4.3 For integers $k \geq 2$ and $1 \leq i \leq k$,

$$\sum_{n_1, \dots, n_{k-1} \geq 0} \frac{q^{N_1^2 + 2N_2^2 + \dots + 2N_{k-1}^2 + 2N_i + \dots + 2N_{k-1}} (-q; q^2)_{N_1}}{(q^2; q^2)_{n_1} \dots (q^2; q^2)_{n_{k-1}}} = \prod_{\substack{j=1 \\ j \not\equiv 2 \pmod{4} \\ j \not\equiv 0, \pm(2i-1) \pmod{4k}}}^{\infty} (1 - q^j)^{-1}. \quad (4.10)$$

Corollary 4.4 For integers $k \geq 2$ and $1 \leq i \leq k - 1$,

$$\begin{aligned} \sum_{n_1, \dots, n_{k-1} \geq 0} \frac{q^{N_1^2 + 2N_2^2 + \dots + 2N_{k-1}^2 + 2N_i + \dots + 2N_{k-1}} (-q; q^2)_{N_1}}{(q^2; q^2)_{n_1} \dots (q^2; q^2)_{n_{k-2}} (q^4; q^4)_{n_{k-1}}} \\ = (-q^{2k-1}, q^{4k-2})_{\infty} \prod_{\substack{j=1 \\ j \not\equiv 2 \pmod{4} \\ j \not\equiv 0 \pmod{8k-4} \\ j \not\equiv 2k-1, \pm(2i-1) \pmod{4k-2}}}^{\infty} (1 - q^j)^{-1}. \end{aligned} \quad (4.11)$$

The identities of corollary 4.3 are due to Andrews [5] and Bressoud [17], and are related to Andrews’ generalization of the Göllnitz–Gordon partition identities [23, 25, 3]. The identities of corollary 4.4 were discovered by Bressoud [17]. These results can also be obtained using the conjugate Bailey pair given by (1.7) and (1.8). For $N \geq 3$, however, theorem 4.1 relies on the higher-level Bailey lemma and, at present, we cannot obtain it by means of the standard conjugate Bailey pair.

Proof of corollaries 4.3 and 4.4

Set $N = 2$ in (4.8) and sum over both choices for σ . Recalling the q -binomial expansion (equation (3.3.6) of ref. [6]) to sum

$$\sum_{n=0}^{r_1 + |\lambda|/2} q^{n(n-|\lambda|)/2} \begin{bmatrix} r_1 + |\lambda|/2 \\ n \end{bmatrix} = (-q^{(1-|\lambda|)/2})_{|\lambda|/2} (-q^{1/2})_{r_1} \quad (4.12)$$

and

$$\sum_{\eta=0}^{\infty} \frac{q^{\eta(\eta-|\lambda|)/2}}{(q)_{\eta}} = (-q^{(1-|\lambda|)/2})_{|\lambda|/2} (-q^{1/2})_{\infty} \quad (4.13)$$

we get, after dropping the factor $(-q^{(1-|\lambda|)/2})_{|\lambda|/2}$, replacing $q \rightarrow q^2$ and applying the triple product identity,

$$\begin{aligned} & \sum_{n_1, \dots, n_{k-1} \geq 0} \frac{q^{N_1^2 + 2N_2^2 + \dots + 2N_{k-1}^2 + 2N_1 + \dots + 2N_{k-1}} (-q; q^2)_{N_1}}{(q^2; q^2)_{n_1} \dots (q^2; q^2)_{n_{k-2}} (q^{4-2\delta}; q^{4-2\delta})_{n_{k-1}}} \\ &= \frac{1}{(q)_\infty} \prod_{n=0}^{\infty} (1 - q^{2+4n})(1 - q^{2i-1+An})(1 - q^{-2i+1+A(n+1)})(1 - q^{A(n+1)}), \end{aligned} \quad (4.14)$$

with $A = 4k + 2\delta - 2$. Rewriting the right-hand side gives equation (4.10) when $\delta = 1$ and equation (4.11) when $\delta = 0$. \square

4.3 Proof of theorem 4.1

The proof of theorem 4.1 is based on the following two Bailey pairs relative to q (equations (2.12) and (2.13) of ref. [7] and item E(3) of ref. [37]):

$$\begin{aligned} \alpha_L &= (-1)^L q^{L(L-1)/2} \frac{(q^2)_{2L}}{(q)_{2L}} \\ \beta_L &= \delta_{L,0} \end{aligned} \quad (4.15)$$

and

$$\begin{aligned} \alpha_L &= (-1)^L q^{L^2} \frac{(q^2)_{2L}}{(q)_{2L}} \\ \beta_L &= \frac{1}{(q^2; q^2)_L}, \end{aligned} \quad (4.16)$$

as well as the Bailey pair (items B(2) and E(4) of ref. [37])

$$\begin{aligned} \alpha_L &= \begin{cases} 1 & L = 0 \\ (-1)^L q^{(\delta+2)L(L-1)/2} (1 + q^{(\delta+2)L}) & L \geq 1 \end{cases} \\ \beta_L &= \frac{q^L}{(q^{2-\delta}; q^{2-\delta})_L}, \end{aligned} \quad (4.17)$$

relative to 1, where $\delta = 0, 1$.

To now derive (4.8) we have to consider various different ranges of i .

Proof of theorem 4.1 for $i = 1$.

We take lemma 4.2 with $a = 1$ and let $\rho_1, \rho_2 \rightarrow \infty$. Hence given a Bailey pair (α, β) relative to 1, the sequences

$$\begin{aligned} \alpha'_L &= q^{L(L+1)} \alpha_L - q^{L(L-1)} (1 - q^{2L}) (\alpha_{L-1} + \dots + \alpha_0) \\ \beta'_L &= q^L \sum_{k=0}^L \frac{q^{k^2} \beta_k}{(q)_{L-k}} \end{aligned} \quad (4.18)$$

form yet again a Bailey pair relative to 1. Now iterating this $k - 2$ times, using (4.17) as initial Bailey pair, we obtain the new Bailey pair

$$\begin{aligned}\alpha_L &= \begin{cases} 1 & L = 0 \\ (-1)^L q^{(2k+\delta-2)L(L-1)/2} \left(1 + q^{(2k+\delta-2)L}\right) & L \geq 1 \end{cases} \\ \beta_L &= q^L \sum_{L \geq r_2 \geq \dots \geq r_{k-1}} \frac{q^{r_2^2 + \dots + r_{k-1}^2 + r_2 + \dots + r_{k-1}}}{(q)_{L-r_2} (q)_{r_2-r_3} \dots (q)_{r_{k-2}-r_{k-1}} (q^{2-\delta}; q^{2-\delta})_{r_{k-1}}}. \end{aligned} \quad (4.19)$$

Substitution into equation (4.1) with $a = 1$ yields theorem 4.1 for $i = 1$. \square

Proof of theorem 4.1 for $2 \leq i \leq k + \delta - 1$.

We take the Bailey pair (4.15) when $\delta = 1$ and (4.16) when $\delta = 0$. Now we apply transformation (1.2) $k - i + \delta - 1$ times, then (4.5) once and then (1.2) $i - 2$ times, all with $\rho_1, \rho_2 \rightarrow \infty$. This yields the following Bailey pair relative to 1:

$$\begin{aligned}\alpha_L &= \begin{cases} 1 & L = 0 \\ (-1)^L q^{((2k+\delta-2)L-2k+2i-\delta)L/2} \left(1 + q^{(2k-2i+\delta)L}\right) & L \geq 1 \end{cases} \\ \beta_L &= \sum_{L \geq r_2 \geq \dots \geq r_{k-1}} \frac{q^{r_2^2 + \dots + r_{k-1}^2 + r_i + \dots + r_{k-1}}}{(q)_{L-r_2} (q)_{r_2-r_3} \dots (q)_{r_{k-2}-r_{k-1}} (q^{2-\delta}; q^{2-\delta})_{r_{k-1}}}, \end{aligned} \quad (4.20)$$

for $i = 2, \dots, k + \delta - 1$. Substituting this into equation (4.1) with $a = 1$ completes the proof. \square

Proof of theorem 4.1 for $3 \leq i \leq k$.

Again we take the Bailey pair (4.15) when $\delta = 1$ and (4.16) when $\delta = 0$. This time we apply transformation (1.2) $k - i + \delta$ times, then (4.3) once and then (1.2) $i - 3$ times, all with $\rho_1, \rho_2 \rightarrow \infty$. This again yields the Bailey pair (4.20) but now for $i = 3, \dots, k + \delta$. Since the cases $i = k$ and $i = k + \delta$ coincide, we only need to keep $i = 3, \dots, k$. After substituting this into equation (4.1) with $a = 1$ we are done. \square

5 Conclusion

We end this paper with a few further remarks about the higher-level Bailey lemma.

In this paper we considered the problem of generalizing the conjugate Bailey pair (1.7)–(1.8), and more specifically its specialization (1.10). Now the very problem of what exactly is meant by “generalization” deserves some discussion. In particular we note the following lemma.

Lemma 5.1 *Let (α, β) and (α', β') be Bailey pairs relative to a and ab , respectively, with*

$$\alpha'_L = \sum_{k=0}^L P_{L,k} \alpha_k \quad \text{and} \quad \beta'_L = \sum_{k=0}^L Q_{L,k} \beta_k, \quad (5.1)$$

and let (γ, δ) be a conjugate Bailey pair relative to a . Then the sequences

$$\gamma'_L = \sum_{k=L}^{\infty} P_{k,L} \gamma_k \quad \text{and} \quad \delta'_L = \sum_{k=L}^{\infty} Q_{k,L} \delta_k, \quad (5.2)$$

form a conjugate Bailey pair relative to ab^{-1} .

Proof. For compactness, write the equations (1.1) and (1.5) as $\beta_L = \sum_{k=0}^L M_{L,k}(a)\alpha_k$ and $\gamma_L = \sum_{k=L}^{\infty} M_{k,L}(a)\delta_k$. Then from the fact that (α, β) and (α', β') are Bailey pairs, and from equation (5.1), we infer the relation

$$\sum_{k=m}^L Q_{L,k} M_{k,m}(a) = \sum_{k=m}^L M_{L,k}(ab) P_{k,m} \quad m = 0, \dots, L. \quad (5.3)$$

With this we compute

$$\begin{aligned} \gamma'_L &= \sum_{k=L}^{\infty} P_{k,L} \gamma_k = \sum_{k=L}^{\infty} P_{k,L} \sum_{m=k}^{\infty} M_{m,k}(a) \delta_m \\ &= \sum_{m=L}^{\infty} \delta_m \sum_{k=L}^m M_{m,k}(a) P_{k,L} = \sum_{m=L}^{\infty} \delta_m \sum_{k=L}^m Q_{m,k} M_{k,L}(a/b) \\ &= \sum_{k=L}^{\infty} M_{k,L}(a/b) \sum_{m=k}^{\infty} Q_{m,k} \delta_m = \sum_{k=L}^{\infty} M_{k,L}(a/b) \delta'_k, \end{aligned} \quad (5.4)$$

establishing the claim of the lemma. \square

The obvious implication of this result is that instead of using the Bailey-chain, Bailey-lattice, et cetera, we could equally well keep (α, β) fixed, and transform (γ, δ) to yield a conjugate Bailey-chain, lattice and so on. Of course, when searching for generalizations of the conjugate Bailey pair (1.10), one must exclude those conjugate Bailey pairs which can simply be obtained from (1.10) by linear transformations of type (5.2). Let us merely state here that the new conjugate Bailey pairs presented in this paper admit no further simplifications, and are, in the loose sense of ref. [7], “reduced”.

The expression on the left-hand side of the higher-level Bailey lemma admits several different representations from the one listed in equation (4.1). The most relevant rewriting arises by first noting that the sum over $\boldsymbol{\eta}$ only depends on $L \pmod{N}$. Using this as well as some other simple symmetries, the left-hand side of equation (4.1) can be put into the form

$$\frac{1}{(aq)_{\infty}} \sum_{p-p\ell=0}^{\lfloor N/2 \rfloor} \sum_{\substack{\boldsymbol{\eta} \in \mathbb{Z}^{N-1} \\ \frac{2p-|\boldsymbol{\lambda}|+\sigma N}{2N} - (C^{-1}\boldsymbol{\eta})_1 \in \mathbb{Z}}} \frac{q^{\boldsymbol{\eta} C^{-1}(\boldsymbol{\eta} - \mathbf{e}_{\lambda})}}{(q)_{\eta_1} \cdots (q)_{\eta_{N-1}}} \sum_{L=0}^{\infty} a^{L/N} q^{L^2/N} \alpha_L, \quad (5.5)$$

where $p_{\ell} = 0$ when ℓ is even and $p_{\ell} = 1/2$ when ℓ is odd. We now assume that the partition λ has just a single part, $\lambda = (\ell')$, with $\ell' = 0, \dots, N-1$. This admits (5.5) to be expressed in terms of a sum over the level- N $A_1^{(1)}$ string functions c_m^{ℓ} [26, 28, 29],

$$q^{-\frac{\ell'(N-\ell')}{2N(N+2)}} (q)_{\ell} \sum_{p-p\ell=0}^{\lfloor N/2 \rfloor} c_{2p+\sigma N}^{\ell'} \sum_{L=0}^{\infty} a^{L/N} q^{L^2/N} \alpha_L, \quad (5.6)$$

$L+\ell/2 \equiv \pm p \pmod{N}$

where $\ell + \ell' + \sigma N$ must be even. An alternative form (to that implied by (5.5)) for c_m^{ℓ} , known as Hecke’s indefinite modular form, reads [26, 28]

$$c_m^{\ell} = \frac{q^{h_m^{\ell}}}{(q)_{\infty}^3} \left\{ \left(\sum_{j \geq 0} \sum_{k \geq 0} - \sum_{j < 0} \sum_{k < 0} \right) (-1)^j q^{j(j+\ell+m+1)/2+k((j+k)(N+2)+\ell+1)} \right.$$

$$+ \left(\sum_{j>0} \sum_{k \geq 0} - \sum_{j \leq 0} \sum_{k < 0} \right) (-1)^j q^{j(j+\ell-m+1)/2+k((j+k)(N+2)+\ell+1)} \Big\}, \quad (5.7)$$

for $\ell = 0, \dots, N-1$, $\ell-m$ even and $|m| \leq \ell$, with $h_m^\ell = \ell(\ell+2)/[4(N+2)] - m^2/(4N)$. To obtain an expression for c_m^ℓ , when $|m| > \ell$, one may use the symmetries $c_m^\ell = c_{-m}^\ell = c_{m+2N}^\ell = c_{N-m}^{N-\ell}$.

As a final comment and advertisement, we remark that an application of the higher-level Bailey lemma to obtain q -series identities for the characters of the level- N $A_1^{(1)}$ coset conformal field theories, will be given in ref. [14] in collaboration with A. Berkovich and B. M. McCoy.

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