Partial-sum analogues of the Rogers–Ramanujan identities

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Dedicated to Barry McCoy on the occasion of his sixtieth birthday

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Abstract

A new polynomial analogue of the Rogers–Ramanujan identities is proven. Here the product-side of the Rogers–Ramanujan identities is replaced by a partial theta sum and the sum-side by a weighted sum over Schur polynomials.

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1 Introduction

The famous Rogers–Ramanujan identities are given by [16]

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{j=0}^{\infty}\frac{1}{(1-q^{5j+1})(1-q^{5j+4})} \tag{1.1}$$

and

$$1+\sum_{n=1}^{\infty}\frac{q^{n(n+1)}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{j=0}^{\infty}\frac{1}{(1-q^{5j+2})(1-q^{5j+3})} \tag{1.2}$$

for $|q|<1$. In one of his two proofs of these identities, Schur [18] introduced two sequences of polynomials $(e_n)_{n\geq 2}$ and $(d_n)_{n\geq 2}$, where e_n (d_n) is the generating function of partitions with difference between parts at least 2 (and no part equal to 1), and largest part at most $n-2$. The partitions $\{\emptyset,(1),(2),(3),(4),(3,1),(4,1),(4,2)\}$, for example, contribute to $e_6=1+q+q^2+q^3+2q^4+q^5+q^6$ and the partitions $\{\emptyset,(2),(3),(4),(4,2)\}$ contribute to $d_6=1+q^2+q^3+q^4+q^6$.

By standard combinatorial arguments, see e.g., [15, 11], it follows that $e_\infty:=\lim_{n\to\infty}e_n=\text{LHS}(1.1)$ and $d_\infty:=\lim_{n\to\infty}d_n=\text{LHS}(1.2)$. Schur proved the Rogers–Ramanujan identities by showing that these limits also hold when LHS is replaced by RHS. This he achieved by showing that both e_n and d_n satisfy the recurrence

$$x_{n+2}=x_{n+1}+q^n x_n, \tag{1.3}$$

and by solving this recurrence subject to the initial conditions $d_1=0$, $e_1=e_2=d_2=1$ (consistent with the combinatorial definition of e_n and d_n for $n\geq 2$). Specifically, Schur's solution to (1.3) reads

$$e_n=\sum_{j=-\infty}^{\infty}(-1)^j q^{j(5j+1)/2}\begin{bmatrix} n-1 \\ \lfloor (n-5j-1)/2\rfloor\end{bmatrix} \tag{1.4a}$$

$$d_n=\sum_{j=-\infty}^{\infty}(-1)^j q^{j(5j+3)/2}\begin{bmatrix} n-1 \\ \lfloor (n-5j-2)/2\rfloor\end{bmatrix} \tag{1.4b}$$

for $n\geq 1$ and $\lfloor x\rfloor$ denoting the integer part of x. Here the q-binomial coefficients are given by $\begin{bmatrix} n \\ m\end{bmatrix}=(q;q)_n/(q;q)_m(q;q)_{n-m}$ for $0\leq m\leq n$ and zero otherwise, where $(a;q)_n=\prod_{j=0}^{n-1}(1-aq^j)$.

Employing the notation $(a_1,\ldots,a_k;q)_n=(a_1;q)_n\cdots(a_k;q)_n$ and recalling to the Jacobi triple product identity [11, Eq. (II.28)]

$$\sum_{n=-\infty}^{\infty}(-1)^n a^n q^{\binom{n}{2}}=(a,q/a,q;q)_\infty \tag{1.5}$$

it is now easy to obtain the desired limits;

$$e_\infty = \frac{1}{(q;q)_\infty} \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2} = \frac{1}{(q,q^4;q^5)_\infty} = \text{RHS}(1.1)$$

$$d_\infty = \frac{1}{(q;q)_\infty} \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+3)/2} = \frac{1}{(q^2,q^3;q^5)_\infty} = \text{RHS}(1.2).$$

Representations for the Schur polynomials similar to the left sides of the Rogers–Ramanujan identities are also known [15, §286 and §289],

$$e_n = \sum_{r=0}^{\infty} q^{r^2} \begin{bmatrix} n-r-1 \\ r \end{bmatrix} \quad \text{and} \quad d_n = \sum_{r=0}^{\infty} q^{r(r+1)} \begin{bmatrix} n-r-2 \\ r \end{bmatrix}. \tag{1.6}$$

Equating this with (1.4) yields the following polynomial analogue of the Rogers–Ramanujan identities [1]:

$$\sum_{r=0}^{\infty} q^{r(r+a)} \begin{bmatrix} n-r-a \\ r \end{bmatrix} = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+2a+1)/2} \begin{bmatrix} n \\ \lfloor (n-5j-a)/2 \rfloor \end{bmatrix} \tag{1.7}$$

for $n \geq 0$ and $a \in \{0,1\}$.

Recently there has been renewed interest in the Schur polynomials [10, 7, 12, 8, 21] sparked by the following nice generalization of the Rogers–Ramanujan identities due to Garrett, Ismail and Stanton [10, Eq. (3.5)]

$$\sum_{r=0}^{\infty} \frac{q^{r(r+m)}}{(q;q)_r} = \frac{(-1)^m q^{-\binom{m}{2}} d_m}{(q,q^4;q^5)_\infty} - \frac{(-1)^m q^{-\binom{m}{2}} e_m}{(q^2,q^3;q^5)_\infty}, \tag{1.8}$$

where m is a nonnegative integer and $e_0 = 0$ and $d_0 = 1$ consistent with (1.3).

In this paper we show that (1.8) may be used to prove new polynomial analogues of the Rogers–Ramanujan identities involving the Schur polynomials. These polynomial identities are fundamentally different from (1.7) in that the product-side is replaced by a partial theta series.

Theorem 1.1. *For $k \in \{0,1\}$ and $n \geq 0$ there holds*

$$\sum_{j=-n-k}^{n} (-1)^j q^{j(5j+1)/2} = \sum_{r=0}^{n} e_{2r+k+2} (-1)^{n-r} q^{(n-r)(5n+3r+4k+5)/2} \frac{(q;q)_{n+r+k}}{(q;q)_{n-r}} \tag{1.9}$$

and

$$\sum_{j=-n-k}^{n} (-1)^j q^{j(5j+3)/2}$$
$$= \sum_{r=0}^{n} d_{2r+k+2} (-1)^{n-r} q^{(n-r)(5n+3r+4k+5)/2} \frac{(q;q)_{n+r+k}}{(q;q)_{n-r}}. \qquad (1.10)$$

Partial theta-sum identities of this type were first discovered by Shanks [19].

When n tends to infinity (for $|q| < 1$) only the term with $r = n$ contributes to the sums on the right. Hence the first identity of the theorem implies

$$\sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2} = (q;q)_\infty e_\infty = (q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n},$$

which is transformed into (1.1) by the triple product identity. Likewise, (1.2) arises as the the large n limit of the second identity of the theorem.

Polynomial analogues of the Rogers–Ramanujan strikingly similar to those of Theorem 1.1 have previously been discovered by Andrews [5]. For $n \geq 0$ let $K_n(x)$ denote the Szegö polynomial [20]

$$K_n(x) = \sum_{r=0}^{n} x^r q^{r(r+1)} \begin{bmatrix} n \\ r \end{bmatrix}.$$

Then Andrews posed in the problems section of SIAM Review [5] the problem of showing that

$$\sum_{j=-n-k}^{n} (-1)^j q^{j(5j+2k+1)/2}$$
$$= \sum_{r=0}^{n} K_r(q^{2n-2r+k-1}) (-1)^{n-r} q^{(n-r)(5n-3r+4k+1)/2} \frac{(q;q)_{n+k}}{(q;q)_{n-r}} \qquad (1.11)$$

for $k = \{0, 1\}$ and $n \geq 0$. Note here that the left side of (1.11) coincides with the left side of (1.9) ((1.10)) when $k = 0$ ($k = 1$).

The remainder of this paper is divided in two parts with section 2 containing a proof and section 3 a discussion of Theorem 1.1. In the first part of this discussion we examine two simple proofs of (1.11) found by Jordan and Andrews and indicate our failure in generalizing these to a proof Theorem 1.1. The second part of our discussion focuses on some of the combinatorial aspects of Theorem 1.1.

2 Proof of Theorem 1.1

2.1 A more general identity

Key to the proof of Theorem 1.1 is the following proposition.

Proposition 2.1. *For $k \in \{0,1\}$ and $|a|,|q|<1$ there holds*

$$\sum_{n=0}^{\infty} \frac{a^{2n} q^{n(n+k)}}{(q;q)_n} = \frac{(a;q)_\infty^2}{(q;q)_\infty^3} \sum_{j=1}^{\infty} (q^{2j-k}, q^{5+k-2j}, q^5; q^5)_\infty \times \sum_{r=0}^{\infty} \frac{(-1)^{j+r+1} q^{\binom{j+r}{2}} (1-q^{2r+k+1})(aq^{-r};q)_r}{(a;q)_{r+k+1}}. \tag{2.1}$$

It is perhaps not immediately clear that the sums on the right converge, but inspection of the potentially problematic terms shows that for $k \in \{0,1\}$ and $j \geq 1$,

$$O\Big(q^{\binom{j+r}{2}} (q^{5+k-2j}; q^5)_\infty (aq^{-r};q)_r\Big) = \begin{cases} q^{(j-1)(j+10r+4k+6)/10} & j \equiv 1, k+4 \pmod 5 \\ q^{(j-2)(j+10r+4k+7)/10+r+1} & j \equiv 2, k+3 \pmod 5, \end{cases}$$

which shows that both sums on the right converge and that their order is irrelevant.

Before proving Proposition 2.1 we will show how it implies Theorem 1.1. Starting point is the observation that

$$\sum_{j=1}^{\infty} (-1)^j q^{\binom{j+r}{2}} \frac{(q^{2j-k}, q^{5+k-2j}, q^5; q^5)_\infty}{(q;q)_\infty} = \sum_{i=1}^{2} \frac{(-1)^{i+r} q^{-\binom{2r+k+2}{2}}}{(q^{i+2k}, q^{5-i-2k}; q^5)_\infty} \sum_{j=-r-k}^{r} (-1)^j q^{j(5j+2i+4k-5)/2}. \tag{2.2}$$

To prove this we use that for f_j such that $f_j = 0$ if $j \equiv 3k \pmod 5$ there holds

$$\sum_{j=1}^{\infty} f_j = \sum_{i=1}^{2} \sum_{j=0}^{\infty} (f_{5j+i} + f_{5j+5-i+k}).$$

This, together with the simple to verify identities

$$(q^{m+5n},q^{5-5n-m};q^5)_\infty=(q^m,q^{5-m};q^5)_\infty(-1)^n q^{-nm-5\binom{n}{2}}$$

$$\frac{(q^{2i-k},q^{5+k-2i},q^5;q^5)_\infty}{(q;q)_\infty}=\frac{1}{(q^{i+2k},q^{5-i-2k};q^5)_\infty},\ i,k+1\in\{1,2\}$$

$$\binom{i+r}{2}-(r+1)(5r+2i+4k)/2=-\binom{2r+k+2}{2},\ i,k+1\in\{1,2\}$$

$$\sum_{i=1}^{2}(-1)^i=0$$

and the Jacobi triple product identity (1.5), yields

$$\begin{aligned}
&\text{LHS(2.2)}\\
&=\sum_{i=1}^{2}\frac{(-1)^i q^{\binom{i+r}{2}}}{(q^{i+2k},q^{5-i-2k};q^5)_\infty}\Biggl(\sum_{j=-\infty}^{-2r-k-2}+\sum_{j=0}^{\infty}\Biggr)(-1)^j q^{j(5j+10r+2i+4k+5)/2}\\
&=\sum_{i=1}^{2}\frac{(-1)^{i+r}q^{\binom{i+r}{2}-(r+1)(5r+2i+4k)/2}}{(q^{i+2k},q^{5-i-2k};q^5)_\infty}\\
&\qquad\times\Biggl[\sum_{j=-r-k}^{r}(-1)^j q^{j(5j+2i+4k-5)/2}-(q^{i+2k},q^{5-i-2k},q^5;q^5)_\infty\Biggr]\\
&=\text{RHS(2.2)}.
\end{aligned}$$

After substituting (2.2) in (2.1) we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{a^{2n}q^{n(n+k)}}{(q;q)_n}\\
&\quad=\frac{(a;q)_\infty^2}{(q;q)_\infty^2}\sum_{r=0}^{\infty}\sum_{i=1}^{2}\frac{(-1)^{i+1}q^{-\binom{2r+k+2}{2}}(1-q^{2r+k+1})(aq^{-r};q)_r}{(q^{i+2k},q^{5-i-2k};q^5)_\infty(a;q)_{r+k+1}}\\
&\qquad\qquad\times\sum_{j=-r-k}^{r}(-1)^j q^{j(5j+2i+4k-5)/2}.
\end{aligned}$$

Here the reader is warned that the order of the sums over r and i must be strictly adhered to. Indeed, our earlier considerations about convergence and the fact that (2.2) is true, guarantee the not so obvious fact that after summing over i the sum over r converges.

Our next step removes any further convergence issues as we now specialize $a=q^{m+1}$ with m a nonnegative integer. The sum over r

then terminates, yielding

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2m+k+2)}}{(q;q)_n} = \sum_{i=1}^{2} \frac{(-1)^{i+1}}{(q^{i+2k}, q^{5-i-2k}; q^5)_\infty} \\ \times \sum_{r=0}^{m} \frac{q^{-\binom{2r+k+2}{2}}(1-q^{2r+k+1})}{(q;q)_{m-r}(q;q)_{m+r+k+1}} \sum_{j=-r-k}^{r} (-1)^j q^{j(5j+2i+4k-5)/2}. \quad (2.3)$$

Rewriting the left-hand side using the Garrett–Ismail–Stanton identity (1.8) gives

$$\frac{d_{2m+k+2}}{(q,q^4;q^5)_\infty} - \frac{e_{2m+k+2}}{(q^2,q^3;q^5)_\infty} = (-1)^k q^{\binom{2m+k+2}{2}} \mathrm{RHS}(2.3).$$

Multiplying both sides by $(q;q)_{2m+k+1}$ this is of the form

$$\frac{P(q)}{(q^2,q^3;q^5)_\infty} = \frac{Q(q)}{(q,q^4;q^5)_\infty}$$

with $P(q)$ and $Q(q)$ polynomials. An identity of this type can only be true if $P(q) = Q(q) = 0$, and we infer

$$e_{2m+k+2} = q^{\binom{2m+k+2}{2}} \sum_{r=0}^{m} \frac{q^{-\binom{2r+k+2}{2}}(1-q^{2r+k+1})}{(q;q)_{m-r}(q;q)_{m+r+k+1}} \sum_{j=-r-k}^{r} (-1)^j q^{j(5j+1)/2}$$

$$d_{2m+k+2} = q^{\binom{2m+k+2}{2}} \sum_{r=0}^{m} \frac{q^{-\binom{2r+k+2}{2}}(1-q^{2r+k+1})}{(q;q)_{m-r}(q;q)_{m+r+k+1}} \sum_{j=-r-k}^{r} (-1)^j q^{j(5j+3)/2}$$

for $m \geq 0$. All that remains is to invert these new representations of the Schur polynomials. This is easily done recalling the Bailey transform [6], which states that if

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}} \quad (2.4)$$

then

$$\alpha_n = (1-aq^{2n}) \sum_{r=0}^{n} \frac{(-1)^{n-r} q^{\binom{n-r}{2}} (aq;q)_{n+r-1}}{(q;q)_{n-r}} \beta_r. \quad (2.5)$$

For later reference we remark that a pair of sequences (α, β) that satisfies (2.4) (or, equivalently, (2.5)) is called a Bailey pair relative to a.

Since our expressions for the Schur polynomials take the form (2.4) with $a = q^{k+1}$, we may invoke (2.5) to find the identity claimed in Theorem 1.1.

2.2 Proof of Proposition 2.1

Our proof relies on the following lemma.

Lemma 2.1. *For $k \in \{0,1\}$ and M and n integers there holds*

$$\begin{aligned}
\frac{q^{n(n+2)}}{(q;q)_\infty^3} \sum_{j=1}^{\infty}\sum_{r=0}^{\infty}\sum_{l=0}^{M} & (-1)^{M+j+r+1} q^{\binom{j+r}{2}+\binom{M-l}{2}+\binom{l}{2}+l(r+k+1)} \\
& \times q^{-r(M-l)-n(2j-k)} \frac{(1-q^{(2j-k)(2n+1)})(1-q^{2r+k+1})}{(1-q^{2n+1})(q;q)_{M-l}(q;q)_l} \\
& = \begin{cases} \dfrac{q^{m(m+k)}}{(q;q)_{m-n}(q;q)_{m+n+1}} & M=2m \\ 0 & M=2m+1. \end{cases}
\end{aligned}$$

Here our earlier definition of $(a;q)_n$ is extended to all integers n by $(a;q)_n = (a;q)_\infty/(aq^n;q)_\infty$. Note in particular that $1/(q;q)_n = 0$ for $n < 0$.

Given the triple sum on the left, Lemma 2.1 perhaps appears complicated and not readily applicable. However, in view of (2.4) it is in fact quite useful, and if we multiply boths sides by α_n and then sum n over the nonnegative integers we get

$$\begin{aligned}
\frac{q^{n(n+2)}}{(q;q)_\infty^3} \sum_{n=0}^{\infty}\sum_{j=1}^{\infty}\sum_{r=0}^{\infty}\sum_{l=0}^{M} & \alpha_n (-1)^{M+j+r+1} q^{\binom{j+r}{2}+\binom{M-l}{2}+\binom{l}{2}+l(r+k+1)} \\
& \times q^{-r(M-l)-n(2j-k)} \frac{(1-q)(1-q^{(2j-k)(2n+1)})(1-q^{2r+k+1})}{(1-q^{2n+1})(q;q)_{M-l}(q;q)_l} \\
& = \begin{cases} q^{m(m+k)}\beta_m & M=2m \\ 0 & M=2m+1, \end{cases}
\end{aligned}$$

where (α,β) is a Bailey pair relative to q.

Next we multiply both sides by a^M and sum over M. If on the left we interchange the sums over M and l, shift $M \to M+l$ and then sum over l and M using Euler's q-exponential sum [11, Eq. (II.2)]

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^n q^{\binom{n}{2}}}{(q;q)_n} = (a;q)_\infty$$

this yields

$$\frac{(a;q)_\infty^2}{(q;q)_\infty^3}\sum_{n=0}^\infty\sum_{j=1}^\infty \alpha_n q^{n(n-2j+k+2)}\frac{(1-q)(1-q^{(2j-k)(2n+1)})}{(1-q^{2n+1})}$$
$$\times\sum_{r=0}^\infty \frac{(-1)^{j+r+1}q^{\binom{j+r}{2}}(1-q^{2r+k+1})(aq^{-r};q)_r}{(a;q)_{r+k+1}}=\sum_{n=0}^\infty a^{2n}q^{n(n+k)}\beta_n.$$

We have nearly arrived at (2.1). All that is needed is the following Bailey pair relative to q due to Rogers [17]:

$$\alpha_n=(-1)^n q^{n(3n+1)/2}\frac{(1-q^{2n+1})}{(1-q)}\quad\text{and}\quad \beta_n=\frac{1}{(q;q)_n}.$$

Substituting this, interchanging the sum over n and j (with the above choice for α_n this may indeed be done) and using the triple product identity (1.5) gives (2.1).

Proof of Lemma (2.1). Replacing M by $2m+i$ where $i\in\{0,1\}$, and shifting $l\to l+m+i$ leads to

$$\frac{q^{n(n+2)}}{(q;q)_\infty^3}\sum_{j=1}^\infty\sum_{r=0}^\infty\sum_{l=-m-i}^{m}(-1)^{j+r+1}q^{\binom{j+r}{2}+ir-n(2j-k)+l(l+2r+k+i+1)}$$
$$\times\frac{(1-q^{(2j-k)(2n+1)})(1-q^{2r+k+1})}{(1-q^{2n+1})(q;q)_{m-l}(q;q)_{m+l+i}}=\frac{\delta_{i,0}}{(q;q)_{m-n}(q;q)_{m+n+1}}.\tag{2.6}$$

By the q-Chu–Vandermonde summation [11, Eq. (II.6)]

$$\sum_{j=0}^n\frac{(a,q^{-n};q)_j\,q^j}{(q,c;q)_j}=\frac{(c/a;q)_n}{(c;q)_n}\,a^n\tag{2.7}$$

this follows from the simpler to prove identity

$$\frac{q^{2n(n+1)}}{(q;q)_\infty^3}\sum_{j=1}^\infty\sum_{r=0}^\infty\sum_{l=-m-i}^{m}(-1)^{j+r+1}q^{\binom{j+r}{2}+ir-n(2j-k)+l(2r+k+1)}$$
$$\times\frac{(1-q^{(2j-k)(2n+1)})(1-q^{2r+k+1})}{(1-q^{2n+1})(q;q)_{m-l}(q;q)_{m+l+i}}=\frac{\delta_{i,0}q^{m-n}}{(q;q)_{m-n}(q;q)_{m+n+1}}.\tag{2.8}$$

Indeed, if we multiply both sides of (2.8) by $q^{m(m+i)}/(q;q)_{M-m}$, the resulting identity can be summed over m by the $c=0$ instance of (2.7) (after first replacing $m\to M-m$). On the right we of course only need to do this sum when $i=0$. Replacing M by m then gives (2.6). Those familiar with the concept of a Bailey chain [6] will have recognized that the reduction of (2.6) to (2.8) corresponds to a simplifying (i.e., backwards) iteration along a Bailey chain relative to q^i.

Since (2.8) is of the form (2.5) with $a=q^i$ we can use (2.4) to invert. Hence

$$\begin{aligned}
&\frac{q^{2n(n+1)}}{(q;q)_\infty^3}\sum_{j=1}^\infty\sum_{r=0}^\infty(-1)^{j+r+1}q^{\binom{j+r}{2}+ir-n(2j-k)-(m+i)(2r+k+1)}\\
&\qquad\times\big(1-q^{(2n+1)(2j-k)}\big)\big(1-q^{2r+k+1}\big)\big(1+q^{(2m+i)(2r+k+1)}\big)\\
&=\delta_{i,0}(1-q^{2m})(1-q^{2n+1})\sum_{r=0}^m\frac{(-1)^{m-r}q^{\binom{m-r}{2}+r-n}(q;q)_{r+m-1}}{(q;q)_{m-r}(q;q)_{r-n}(q;q)_{r+n+1}}, \qquad (2.9)
\end{aligned}$$

with the convention that $(1-q^{2m})(q^m;q)_{r+m-1}=2$ for $m=r=0$ in accordance with $(1-q^{2m})(q;q)_{m-1}=(1+q^m)(q;q)_m$. The sum over r on the right may be carried out by the q-Chu–Vandermonde sum (2.7), leading to

$$\begin{aligned}
&\mathrm{RHS}(2.9)\\
&=\delta_{i,0}\frac{(-1)^{m+n}(1-q^{2m})(1-q^{2\max\{n,-n-1\}+1})(q;q)_{m+\max\{n,-n-1\}-1}}{(q;q)_{m-n}(q;q)_{m+n+1}(q^2;q)_{\max\{n,-n-1\}-m}},
\end{aligned}$$

which is nonzero for $n=\pm m$ and $n=\pm m-1$ only. If we also multiply (2.9) by $q^{i(k+1)/2}$ and note that on the right this may again be dropped, we obtain

$$\begin{aligned}
&\frac{q^{2n(n+1)}}{(q;q)_\infty^3}\sum_{j=1}^\infty\sum_{r=0}^\infty(-1)^{j+r+1}q^{\binom{j+r}{2}-n(2j-k)-(2m+i)(2r+k+1)/2}\\
&\qquad\times\big(1-q^{(2n+1)(2j-k)}\big)\big(1-q^{2r+k+1}\big)\big(1+q^{(2m+i)(2r+k+1)}\big)\\
&\qquad\qquad=\delta_{i,0}\big(\delta_{m,n}+\delta_{-m,n}-\delta_{m-1,n}-\delta_{-m-1,n}\big). \qquad (2.10)
\end{aligned}$$

Since both sides are invariant under the substitution $m\to-m-i$ this must hold for all $m,n\in\mathbb{Z}$ and $i,k\in\{0,1\}$.

Next we observe that (2.10) is a consequence of the stronger result

$$\begin{aligned}
&\frac{q^{\binom{n}{2}+(n-m)/2}}{(q;q)_\infty^3}\sum_{j=1}^\infty\sum_{r=0}^\infty(-1)^{j+r+n}q^{\binom{j+r}{2}-(n-1)(2j-k)/2-(m-1)(2r+k)/2}\\
&\qquad\times\big(1-q^{n(2j-k)}\big)\big(1-q^{m(2r+k+1)}\big)=\delta_{m,n}-\delta_{-m,n}, \qquad (2.11)
\end{aligned}$$

for $m,n\in\mathbb{Z}$ and $k\in\{0,1\}$. If we denote the above two identities by $(2.10)|_{m,n}$ and $(2.11)|_{m,n}$ and note that $\delta_{2m+i\pm1,2n+1}=\delta_{i,0}\delta_{m-(1\mp1)/2,n}$, then $(2.10)|_{m,n}=(2.11)|_{2m+i+1,2n+1}-(2.11)|_{2m+i-1,2n+1}$.

Before proving (2.11) let us point out that without loss of generality we may fix $k=0$. For, if we take (2.11) with $k=1$, replace $r\leftrightarrow j-1$, and multiply the result by $(-1)^{m-n}q^{(m^2-n^2)/2}$ we find $(2.11)|_{m,n;k=1}=(2.11)|_{n,m;k=0}$. Equation (2.11) for $k=0$ is a linear combination of yet another identity, given by

$$\begin{aligned}&\frac{q^{\binom{n}{2}}}{(q;q)_\infty^3}\sum_{j=0}^{\infty}\sum_{r=0}^{\infty}(-1)^{j+r+n}q^{\binom{j+r}{2}-(n-1)j-(m-1)r}\\&+\frac{q^{\binom{n}{2}}}{(q;q)_\infty^3}\sum_{j=1}^{\infty}\sum_{r=0}^{\infty}(-1)^{j+r+n}q^{\binom{j+r}{2}-(n-1)j-(m-1)r+2nj+m(2r+1)}=\delta_{m,n}.\end{aligned}\tag{2.12}$$

Here it should be noted that the first sum over j now includes the term $j=0$. It is easily seen that this extra term is cancelled out in the following linear combination, and that $(2.12)|_{m,n}-q^{-n}(2.12)|_{m,-n}=q^{(m-n)/2}(2.11)|_{m,n;k=0}$.

After this string of reductive steps we are finally in a position to carry out a proof. Replacing m by $m+n$ in (2.12) and changing the summation variable $j\to n-j$ ($j\to j-n$) in the first (second) double sum gives

$$\sum_{j=-\infty}^{n}\sum_{r=0}^{\infty}(-1)^{j+r}q^{\binom{j-r}{2}-mr}+\sum_{j=n+1}^{\infty}\sum_{r=0}^{\infty}(-1)^{j+r}q^{\binom{j+r+1}{2}+m(r+1)}=(q;q)_\infty^3\delta_{m,0}.$$

In the second term on the left we rewrite the sum over r using

$$\sum_{r=0}^{\infty}(-1)^r q^{\binom{r+1}{2}+a(r+1)}=\sum_{r=0}^{\infty}(-1)^r q^{\binom{r+1}{2}-ar}\tag{2.13}$$

as follows from

$$\sum_{r=-\infty}^{\infty}(-1)^r q^{\binom{r+1}{2}-ar}=0.\tag{2.14}$$

(To prove (2.14) replace $r\to 2a-1-r$.) As a result we are left with

$$\sum_{j=-\infty}^{\infty}\sum_{r=0}^{\infty}(-1)^{j+r}q^{\binom{j-r}{2}-mr}=(q;q)_\infty^3\delta_{m,0}.\tag{2.15}$$

Using (2.13) on the sum over r and negating j yields $(2.15)|_m = q^m(2.15)|_{-m}$, so that we may assume $m \le 0$ when proving (2.15). If $m < 0$ the order of the sums may be interchanged. By (2.14) this completes the proof. If $m = 0$ we need

$$\sum_{r=0}^{2j-1}(-1)^r q^{\binom{j-r}{2}} = 0$$

for $j \ge 0$ (to prove this replace $r \to 2j-1-r$), and Jacobi's identity [13, §66, (5.)]

$$\sum_{i=0}^{\infty}(-1)^i(2i+1)q^{\binom{i+1}{2}} = (q;q)_\infty^3.$$

Equipped with these the rest is easy;

$$\begin{aligned}\sum_{j=-\infty}^{\infty}\sum_{r=0}^{\infty}(-1)^{j+r}q^{\binom{j-r}{2}} &= \Big\{\sum_{j=0}^{\infty}\sum_{r=2j}^{\infty} + \sum_{j=-\infty}^{-1}\sum_{r=0}^{\infty}\Big\}(-1)^{j+r}q^{\binom{j-r}{2}} \\ &= \Big\{\sum_{j=0}^{\infty} + \sum_{j=1}^{\infty}\Big\}\sum_{r=j}^{\infty}(-1)^r q^{\binom{r+1}{2}} \\ &= \sum_{r=0}^{\infty}(-1)^r(2r+1)q^{\binom{r+1}{2}} = (q;q)_\infty^3. \qquad \square\end{aligned}$$

3 Discussion

3.1 Eqs (1.9) and (1.10) versus (1.11)

The proof of Theorem 1.1 as given in the previous section is very lengthy and complicated, and, as a result, not very illuminating. Here we briefly discuss the proofs of (1.11) as found by Jordan and Andrews as we hold some hope that at least one of these may be generalized to also prove (1.9) and (1.10).

Perhaps simplest is Jordan's proof [14]. Denoting the right side of (1.11) by $f_{n;k}$ and the summand on the right of (1.11) by $f_{n,r;k}$, it is not difficult to show that the functional equation

$$(1 - xq^{n+2})K_{n+1}(x) = K_n(x) - x^2q^{n+4}K_n(xq^2)$$

satisfied by the Szegö polynomials implies the recurrence

$$\sum_{r=m+1}^{n}(f_{n,r;k} - f_{n-1,r-1;k}) = -\frac{1-q^{m-n}}{1-q^{n+k}}f_{n,m;k}. \tag{3.1}$$

By the $m = 0$ instance hereof it is found that

$$\begin{aligned} f_{n;k} - f_{n-1;k} &= f_{n,0;k} + \sum_{r=1}^{n} (f_{n,r;k} - f_{n-1,r-1;k}) \\ &= q^{-n} \frac{1-q^{2n+k}}{1-q^{n+k}} f_{n,0;k} \\ &= (-1)^n q^{n(5n+2k+1)/2} + (-1)^{n+k} q^{(n+k)(5(n+k)-2k-1)/2}, \end{aligned}$$

from which (1.11) follows by induction. Unfortunately, at present we have been unable to find an analogue of (3.1) for the summands on the right of Theorem 1.1.

Andrews' proof of (1.11) relies on the following multiple series generalization of Watson's q-Whipple transform [3, Thm. 4; $k = 3$]:

$$\begin{aligned} &{}_{10}W_9(a; b, c, d, e, f, g, q^{-n}; q, a^3 q^{n+3}/bcdefg; q, q) \\ &\qquad = \frac{(aq, aq/fg; q)_n}{(aq/f, aq/g; q)_n} \sum_{j=0}^{n} \sum_{k=0}^{n-j} \frac{(aq/bc, d, e; q)_j}{(q, aq/b, aq/c; q)_j} \frac{(aq/de; q)_k}{(q; q)_k} \\ &\qquad\qquad \times \frac{(f, g, q^{-n}; q)_{j+k}}{(aq/d, aq/e, fgq^{-n}/a)_{j+k}} \Big(\frac{aq^2}{de}\Big)^j q^k. \end{aligned} \tag{3.2}$$

Here and in the following we employ standard notation for basic hypergeometric series, see e.g., [11]. Taking $b = aq^{n+1}$ and letting c, d, e, f, g tend to infinity yields (1.11) with $k = 0$ if $a = 1$ ($k = 1$ if $a = q$).

Now if we apply Sears' ${}_4\phi_3$ transformation [11, Eq. (III.15)] to Watson's Watson's q-Whipple transform [11, Eq. (III.18)] we readily obtain

$$\begin{aligned} &{}_8W_7(a; b, c, d, e, q^{-n}; q, a^2 q^{n+2}/bcde; q, q) \\ &\qquad = \frac{(aq, b, a^2q^2/bcde; q)_n}{(aq/c, aq/d, aq/e; q)_n} {}_4\phi_3 \left[\begin{matrix} aq/bc, aq/bd, aq/be, q^{-n} \\ aq/b, a^2q^2/bcde, q^{1-n}/b \end{matrix} ; q, q \right]. \end{aligned} \tag{3.3}$$

Taking $b = aq^{n+1}$ and letting c, d, e, f, g this simplifies to

$$\sum_{j=0}^{n} \frac{1-aq^{2j}}{1-a} \frac{(a;q)_j (-1)^j q^{3\binom{j}{2}} (aq)^j}{(q;q)_j} = (aq;q)_{2n} \sum_{j=0}^{n} \frac{q^j}{(q, q^{-2n}/a; q)_j}. \tag{3.4}$$

Chosing $a = q^k$ for $k = 0, 1$ and making the variable change $j \to n - r$ on the right we obtain the following polynomial analogue of Euler's identity

$$\sum_{j=-n-k}^{n}(-1)^j q^{j(3j+1)/2}=\sum_{r=0}^{n}(-1)^{n-r}q^{(n-r)(3n+r+2k+3)/2}\frac{(q;q)_{n+r+k}}{(q;q)_{n-r}}. \tag{3.5}$$

(Incidentally, this identity is very similar and can easily be transformed into a polynomial version of Euler's identity due to Shanks [19].) Given the similarity between (3.5) and the identities (1.9) and (1.10), and given Andrews' proof of (1.11) by means of (3.2) it seems very natural to ask for a proof of Theorem 1.1 by means of a multiple series generalization of the transformation (3.3). If we take (1.9) with $k=0$ and (1.10) with $k=1$ and replace $r\to n-j$ in the sums on the right we find that the resulting identities are the $a=1$ and $a=q$ instances of

$$\begin{aligned}\sum_{j=0}^{n}&\frac{1-aq^{2j}}{1-a}\frac{(a;q)_j(-1)^j q^{5\binom{j}{2}}(aq)^{2j}}{(q;q)_j}\\&=(aq;q)_{2n}\sum_{j=0}^{n}\sum_{k=0}^{n-j}\frac{(-1)^k a^{j+k}q^{j^2-\binom{2j+k}{2}+(2n+1)(j+k)}(q^{-2n-1};q)_{2j+2k}}{(q,q^{-2n}/a;q)_j(q;q)_k(q^{-2n-1};q)_{2j+k}}\end{aligned}$$

This is to be compared with (3.4). Despite numerous attempts we failed to extend this to a multiple series transformation similar to (3.2) and generalizing (3.3). Of course one can try to prove the above by equating coefficients of a^m, but the resulting identity

$$\begin{aligned}\sum_{j=0}^{n}&\frac{(-1)^j q^{\binom{j}{2}+j(4j-2m+1)}}{(q;q)_j}\left(\begin{bmatrix}j\\m-2j\end{bmatrix}+q^{4j-m+1}\begin{bmatrix}j\\m-2j-1\end{bmatrix}\right)\\&=\sum_{r=0}^{n}\sum_{s=0}^{n-r}\frac{(-1)^{r+s}q^{\binom{r+s}{2}+r(4n-2m+4)+s(s+r-m+1)}}{(q;q)_r}\\&\qquad\times\begin{bmatrix}2n+1-r\\m-2r-s\end{bmatrix}\begin{bmatrix}2n-2r-s+1\\s\end{bmatrix}\end{aligned}$$

for $n\geq 0$ and $0\leq m\leq 3n+1$ is not particularly simple (and would only prove half of Theorem 1.1).

3.2 Some combinatorics related to Theorem 1.1

In order to discuss some of the combinatorics of Theorem 1.1 we need to review several standard results from partition theory [4].

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition, defined as a weakly decreasing sequence of positive integers λ_j (the parts of λ). The weight $|\lambda|$ of λ is given by the sum of its parts. We say that λ is a partition of l if $|\lambda| = l$. The Ferrers graph of λ is the graph obtained by drawing r left-aligned rows of dots with the jth row containing λ_j dots. The conjugate λ' of λ is obtained by transposing its Ferrers graph. The number $d(\lambda)$ is the number of rows in the maximal square of dots of the Ferrers graph of λ. An alternative way to represent a partition λ is as a two-rowed matrix of $d(\lambda) = d$ columns $\binom{t_1 t_2 \ldots t_d}{b_1 b_2 \ldots b_d}$, where $t_j = \lambda_j - j$ and $b_j = \lambda'_j - j$, so that, in particular, $t_j > t_{j+1}$ and $b_j > b_{j+1}$. Conversely, any such matrix (also called Frobenius symbol) corresponds to the unique partition λ by $\lambda_j = t_j + b_j + 1$. We will in the following identify the standard and Frobenius notations for partitions. Note that $|\lambda| = d + \sum_{j=1}^{d} (t_j + b_j)$. The rank of a partition λ is defined as its largest part minus its number of parts, i.e., as $\lambda_1 - \lambda'_1 = t_1 - b_1$. More generally, the ith successive rank of λ is given by $t_i - b_i$, and $r(\lambda) = (t_1 - b_1, t_2 - b_2, \ldots, t_d - b_d)$ denotes the sequence of successive ranks of λ. For example, if $\lambda = (7,7,5,3,3,1,1,1)$, then $|\lambda| = 28$, $\lambda' = (8,5,5,3,3,2,2)$, $d(\lambda) = 3$, $\lambda = \binom{652}{732}$, $\lambda' = \binom{732}{652}$, and $r(\lambda) = (-1, 2, 0)$.

Now let $b_2(l,n)$ denote the set of all partitions of l, with largest part at most $n-2$ and difference between parts at least 2, and let $B_2(l,n)$ be its cardinality. Then $e_n = \sum_{l \geq 0} B_2(l,n) q^l$. Given a partition $\lambda \in b_2(l,n)$ with exactly r parts, one can form a new partition μ as follows [4, §9.3]: $\mu = \binom{s_1, \ldots, s_r}{c_1, \ldots, c_r}$, where $s_j = \lfloor \lambda_j / 2 \rfloor$ and $c_j = \lfloor (\lambda_j - 1)/2 \rfloor$. Because of the difference-2 condition one indeed has $s_j > s_{j+1}$ and $c_j > c_{j+1}$. Since (for $n \in \mathbb{Z}$) $\lfloor n/2 \rfloor + \lfloor (n-1)/2 \rfloor = n - 1$ one finds that $|\mu| = r + \sum_{j=1}^{r} (s_j + c_j) = |\lambda| = l$. Furthermore, the restriction that $\lambda_j - \lambda_{j+1} \geq 2$ translates into the fact that the successive ranks of μ must take the values 0 and 1 only. Finally the restriction that $\lambda_1 \leq n-2$ implies that $s_1 + c_1 + 1 \leq n - 2$. Since $s_1 - c_1 \in \{0,1\}$ this is equivalent to requiring that $\mu_1 \leq \lfloor n/2 \rfloor$ and $\mu'_1 \leq \lfloor (n-1)/2 \rfloor$. If we denote the set of all partitions of l with successive ranks in $\{0,1\}$, largest part not exceeding $\lfloor n/2 \rfloor$ and number of parts not exceeding $\lfloor (n-1)/2 \rfloor$ by $q_2(l,n)$ (with cardinality $Q_2(l,n)$) then clearly each partition $\mu \in q_2(l,n)$ can also be mapped back onto a partition in $b_2(l,n)$. Specifically, if $\mu \in q_2(l,n)$ has Frobenius symbol $\binom{s_1, \ldots, s_r}{c_1, \ldots, c_r}$, then $\lambda = (s_1 + c_1 + 1, \ldots, s_r + c_r + 1) \in b_2(l,n)$, since $\lambda_j - \lambda_{j+1} = s_j + c_j - s_{j+1} - c_{j+1} \geq 2$ and $\lambda_1 = s_1 + c_1 + 1 = \mu_1 + \mu'_1 - 1 \leq n - 2$. Hence $Q_2(l,n) = B_2(l,n)$ and $e_n = \sum_{l \geq 0} Q_2(l,n) q^l$. For example, $\cup_{l \geq 0} q_2(l,n) = \{\emptyset, (1), (2), (2,1), (3,1), (2,2), (3,2), (3,3)\}$ so that $e_6 =$

$1+q+q^2+q^3+2q^4+q^5+q^6$.

The above discussion can be repeated for the Schur polynomial d_n and we define $b_1(l,n)$ as the subset of $b_2(l,n)$ obtained by removing all partitions which have a part equal to 1. Hence $d_n = \sum_{l\geq 0} B_1(l,n)q^l$. If we also define $q_1(m,n)$ as the set of partitions with successive ranks in $\{1,2\}$, largest part not exceeding $\lfloor (n+1)/2 \rfloor$ and number of parts not exceeding $\lfloor (n-2)/2 \rfloor$, then it is not hard to show that $Q_1(l,n) = B_1(l,n)$ so that $d_n = \sum_{l\geq 0} Q_1(l,n)q^l$. For example, $\cup_{l\geq 0} q_1(l,n) = \{\emptyset, (2), (3), (3,1), (3,3)\}$ so that $d_6 = 1+q^2+q^3+q^4+q^6$.

So far, we have given a combinatorial interpretation of the Schur polynomials e_n and d_n in terms of partitions with restrictions on their size and successive ranks. Next we will discuss the combinatorial interpretation of the partial theta sum $\sum_{j=-n+k}^{n} (-1)^j q^{j(5j-2i+5)/2}$ in terms of successive ranks.

First we recall some further known properties of $Q_i(l,n)$ [2, 4]. Let λ be a partition and $r(\lambda)$ its sequence of successive ranks. The length of the largest subsequence r' of $r(\lambda)$ such that the odd (even) elements of r' are at least $4-i$ and the even (odd) elements of r' are at most $1-i$, is called the $(2,i)$-positive ($(2,i)$-negative) oscillation of λ. The number of partitions of l that have at most b parts, largest part not exceeding a and $(2,i)$-positive ($(2,i)$-negative) oscillation at least j is denoted by $p_i(a,b;j;l)$ ($m_i(a,b;j;l)$). By inclusion-exclusion arguments it then follows that

$$Q_i(l,n) = \sum_{j=0}^{\infty} (-1)^j p_i(\bar{a},\bar{b};j,l) + \sum_{j=1}^{\infty} (-1)^j m_i(\bar{a},\bar{b};j,l),$$

with $\bar{a} = \bar{a}(n,i) = \lfloor (n-i+2)/2 \rfloor$ and $\bar{b} = \bar{b}(n,i) = \lfloor (n+i-3)/2 \rfloor$. Furthermore,

$$q^{j(5j-2i+5)/2} \begin{bmatrix} n-1 \\ \lfloor \frac{n+i-5j-3}{2} \rfloor \end{bmatrix} = \begin{cases} \sum_{l=0}^{\infty} p_i(\bar{a},\bar{b};-j,l)q^l & j\leq 0,\ j \text{ even} \\ \sum_{l=0}^{\infty} m_i(\bar{a},\bar{b};-j,l)q^l & j\leq 0,\ j \text{ odd} \\ \sum_{l=0}^{\infty} p_i(\bar{a},\bar{b};j,l)q^l & j\geq 0,\ j \text{ odd} \\ \sum_{l=0}^{\infty} m_i(\bar{a},\bar{b};j,l)q^l & j\geq 0,\ j \text{ even}, \end{cases}$$

from which (1.4) immediately follows. But now it is also clear what our partial theta sums represent. If we denote by $\lambda_{i,j}^{\pm}$ the (unique) partition of minimal weight that has a positive/negative $(2,i)$-oscillation j and by M_i the set of all such minimal partitions, i.e., $M_i = \{\lambda_{i,j}^{\sigma}\}_{j\geq 0;\sigma\in\{0,1\}}$, then (for $k \in \{0,1\}$ and $i \in \{1,2\}$)

$$\sum_{j=-n-k}^{n} (-1)^j q^{j(5j-2i+5)/2} = \sum_{\substack{\lambda \in M_i \\ \lambda_1 \le \lfloor (5n+2ki-2i+5)/2 \rfloor \\ \lambda_1' \le \lfloor (5n+2ki)/2 \rfloor}} (-1)^{d(\lambda)} q^{|\lambda|}.$$

One can in fact easily find the partition $\lambda^{\pm}_{i,j}$. For example, using the Frobenius notation it follows immediately that for j even

$$\lambda^+_{i,j} = \begin{pmatrix} 5j/2-1, & 5j/2-5, & \dots, & 9, & 5, & 4, & 0 \\ 5j/2+i-5, & 5j/2+i-6, & \dots, & i+5, & i+4, & i, & i-1 \end{pmatrix}.$$

When converted into standard notation this gives

$$\lambda^+_{i,j} = (5j/2, (5j/2-3)^2, \dots, (j+6)^2, (j+3)^2, j^i, (j-2)^3, \dots, 4^3, 2^3)$$

where p^f stands for f parts of size p. Calculating the weight of this partition gives

$$|\lambda^+_{i,j}| = 5j/2 + 2\sum_{k=1}^{j/2-1}(j+3k) + ij + 3\sum_{k=1}^{j/2-1}(2k) = j(5j+2i-5)/2, \quad j \text{ even}$$

as it should. Similarly one can use the Frobenius notation to find

$$\lambda^-_{i,j} = ((5j-3)/2)^2, \dots, (j+6)^2, (j+3)^2, j^i, (j-2)^3, \dots, 3^3, 1^3),$$

$(\lambda^+_{i,j})' = \lambda^-_{5-i,j}$ both for j odd, and $(\lambda^-_{i,j})' = \lambda^+_{5-i,j}$ for j even, and thus $|\lambda^{\pm}_{i,j}| = j(5j \mp 2i \pm 5)/2$ for odd j and $|\lambda^-_{i,j}| = j(5j-2i+5)/2$ for even j.

Summarizing, we have the following remarkable situation. The Schur polynomials, which are the generating functions of certain size and successive rank restricted partitions, can be expressed as an alternating sum over the generating functions of partitions with certain restrictions on their $(2,i)$-oscillations. This well-known fact [2, 4] provides a combinatorial explanation of Schur's result (1.4). But now we see that according to the Theorem 1.1 there is another side to the coin; the alternating sum over the generating function of a very special subset of partitions with certain restrictions on their $(2,i)$-oscillations can in its turn be expressed as a weighted sum over Schur polynomials. However, by no means is this an example of a trivial (or nontrivial but known) inversion result. Indeed, naively one might think that if we substitute (1.4) in Theorem 1.1 to get

$$\sum_{j=-n-k}^{n} (-1)^j q^{j(5j-2i+5)/2} = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j-2i+5)/2} \times \sum_{r=0}^{n} (-1)^{n-r} q^{(n-r)(5n+3r+4k+5)/2} \frac{(q;q)_{n+r+k}}{(q;q)_{n-r}} \begin{bmatrix} 2r+k+1 \\ \lfloor (2r+i+k-5j-1)/2 \rfloor \end{bmatrix},$$

that this is just a consequence of the second line being $\chi(-n-k \le j \le n)$ with $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$. However, it is readily checked that this is only correct when $n = k = 0$. It thus seems an extremely challenging problem to find a combinatorial proof of Theorem 1.1, especially since our analytic proof provides so little insight as to why this theorem is true.

To conclude we remark that the previous discussion has a representation theoretic counterpart. As is well-known,

$$\frac{1}{(q;q)_\infty} \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j-2i+5)/2}$$

is the (normalized) character of the $c = -22/5$ Virasoro algebra corresponding to the highest weight vector v_{h_i} of weight $h_i = (1-i)/5$. According to the Feigin–Fuchs construction [9] the above character can be constructed from the Verma module $V(c, h_i)$ by eliminating submodules generated by singular or null vectors. Because of the embedded structure of these submodules this leads to an inclusion-exclusion type of sum. Specifically, the character corresponding to the submodule $V(c, h_i')$ with singular vector of weight h_i' is given by $q^{h_i'-h_i}/(q;q)_\infty$, with the set of weights of singular vectors (including v_{h_i}) given by $h_i' = h_i + j(j-2i+5)/2$ for $j \in \mathbb{Z}$. Therefore, if we denote by $V_s(c, h_i; N)$ the set comprising of the N singular vectors of $V(c, h_i)$ of smallest weight, and if we denote by $d(v) + 1$ the number of (sub)modules $V(c, h_i')$ that contain the singular vector v (so that $d(v) = 0$ iff $v = v_{h_i}$), then

$$\sum_{j=-n-k}^{n} (-1)^j q^{j(5j-2i+5)/2} = \sum_{v \in V_s(c,h_i;2n+k+1)} (-1)^{d(v)} q^{|v|-h_i},$$

where $|v|$ is the weight of v, $c = -22/5$ and $h_i = (1-i)/5$. Again it is a challenge to explain Theorem 1.1 from the above representation theoretic point of view.

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