

Supernomial coefficients, polynomial identities and q -series

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Abstract

q -Analogues of the coefficients of x^a in the expansion of $\prod_{j=1}^N (1 + x + \cdots + x^j)^{L_j}$ are proposed. Useful properties, such as recursion relations, symmetries and limiting theorems of the “ q -supernomial coefficients” are derived, and a combinatorial interpretation using generalized Durfee dissection partitions is given. Polynomial identities of boson–fermion-type, based on the continued fraction expansion of p/k and involving the q -supernomial coefficients, are proven. These include polynomial analogues of the Andrews–Gordon identities. Our identities unify and extend many of the known boson–fermion identities for one-dimensional configuration sums of solvable lattice models, by introducing multiple finitization parameters.

1 Introduction

The Gaussian polynomial or q -binomial coefficient is defined as

$$\begin{bmatrix} L \\ a \end{bmatrix} = \begin{cases} \frac{(q^{L-a+1})_a}{(q)_a} & \text{for } a \in \mathbb{Z}_+, L \in \mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases} \quad (1.1)$$

where \mathbb{Z}_+ is the set of non-negative integers, $(x)_\infty = \prod_{i=0}^{\infty} (1 - xq^i)$ and $(x)_n = (x)_\infty / (xq^n)_\infty$ for $n \in \mathbb{Z}$ (see for example [28]). Many nice identities involve the q -binomial, one of the simplest

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being

$$(-x)_L = \sum_{a=0}^{\infty} \begin{bmatrix} L \\ a \end{bmatrix} x^a q^{a(a-1)/2}, \quad (1.2)$$

which is the q -analogue of the binomial expansion for $(1+x)^L$. For non-negative L equation (1.2) asserts the combinatorial statement that $q^{a(a-1)/2} \begin{bmatrix} L \\ a \end{bmatrix}$ is the generating function of partitions with at most a parts, no part exceeding $L-1$, and all parts distinct. This result naturally leads to the question whether the multinomial coefficients defined by

$$(1+x+\cdots+x^N)^L = \sum_{a=-\frac{NL}{2}}^{\infty} \begin{pmatrix} L \\ a \end{pmatrix}_N x^{a+\frac{NL}{2}}, \quad (1.3)$$

also admit a q -analogue. (Note our convention that a is half an odd integer when NL is odd.)

For $N=2$ this question was addressed by Andrews and Baxter [9] who defined q -trinomial coefficients and studied some of their properties. Interestingly though, equation (1.3) does not admit a q -analogue, and the combinatorial significance of the q -trinomial coefficients was only subsequently revealed, leading to a new proof of Schur's partition theorem [8].

Recently, q -multinomial coefficients for arbitrary N were introduced [40, 47], generalizing the Andrews–Baxter expressions. Surprisingly, the combinatorial interpretation of these q -multinomials generalizes neither of the just mentioned partition results, but the fact that $q^{a^2} \begin{bmatrix} L \\ a \end{bmatrix}$ is the generating function of partitions with exactly a parts, no part exceeding L and no part less than a . In refs. [40, 47] polynomial identities involving the q -multinomials were proven. These identities can be viewed as finite or polynomial analogues of Rogers–Ramanujan or boson–fermion identities (see section 5.2 for the definition of the latter). That is, for each value of a “finitization parameter” L there is a polynomial identity, such that in the limit $L \rightarrow \infty$ a q -series identity of Rogers–Ramanujan or boson–fermion-type is recovered.

In this paper we generalize the q -multinomial coefficients to “ q -supernomial coefficients”, or in short q -supernomials, by defining the natural q -analogues of the coefficients of x^a in the expansion of $\prod_{j=1}^N (1+x+\cdots+x^j)^{L_j}$. Using the q -supernomials, we prove polynomial identities labelled by not one but multiple finitization parameters. First, a family of polynomial identities related to Andrews' analytic counterpart of the Gordon partition theorem is considered. Subsequently, a much more general class of supernomial identities is proven, based on the continued fraction expansion of p/k . For both cases the proof relies on simple q -supernomial recurrences and on known polynomial identities with a single finitization parameter, acting as initial conditions. For the Andrews–Gordon case these are the polynomial boson–fermion identities of Foda and Quano [25] and Kirillov [34]. For the continued fraction identities, the initial conditions are the polynomial identities of Berkovich and McCoy [14, 17] for finite analogues of the minimal Virasoro characters $\chi^{(p-k,p)}$. Besides the special cases used as initial conditions, our supernomial identities take many of the known boson–fermion polynomial identities for one-dimensional configuration sums of solvable lattice models as special one-variable subcases. Taking the q -series limits of the q -supernomial identities, we obtain new boson–fermion type identities for generalizations of the $A_1^{(1)}$ branching functions.

The first part of this paper, comprised of sections 2 and 3, develops the general theory of q -supernomials. Section 2 is devoted to the definition of the q -supernomials as N -variable generalizations of the q -multinomial coefficients, and to stating its most fundamental properties, such as symmetries, recurrences and q -series limits. In section 3 a combinatorial interpretation

of q -supernomials is given using Durfee dissections. This interpretation is then employed to show how several of the results established in section 2 can be understood purely combinatorially. The second part of this paper deals with the application of q -supernomials to polynomial identities labelled by multiple finitization parameters. The supernomial identities related to the Andrews–Gordon identity are given in section 4. The supernomial identities based on the continued fraction expansion of p/k , and their corresponding q -series identities, are the subject of section 5. We conclude with an outlook on open problems related to the q -supernomials.

2 q -Supernomial coefficients

2.1 Preliminaries

Before defining the q -supernomial coefficients, we introduce some general notation used throughout this paper. q -Supernomials are multivariable generalizations of the q -multinomial coefficients. It is therefore useful to introduce a vector notation, and for a fixed positive integer N , we set $\mathbf{L} = (L_1, \dots, L_N)$, where each of the entries $L_j \in \mathbb{Z}$. We further need the Cartan matrix C of the Lie algebra A_{N-1} , with components $C_{i,j} = 2\delta_{i,j} - \delta_{|i-j|,1}$, and the Cartan-type matrix T , associated with the tadpole graph of N nodes, with components $T_{i,j} = \delta_{i,j}(2 - \delta_{j,N}) - \delta_{|i-j|,1}$. The entries of the inverse of these matrices are given by

$$\begin{aligned} C_{i,j}^{-1} &= \min\{i, j\} - \frac{ij}{N} & i, j &= 1, \dots, N-1, \\ T_{i,j}^{-1} &= \min\{i, j\} & i, j &= 1, \dots, N. \end{aligned} \tag{2.1}$$

Besides \mathbf{L} , it is often convenient to use the vector $\boldsymbol{\ell} = T^{-1}\mathbf{L}$. (Throughout this paper transposition symbols are omitted.) Finally we use the unit vectors \mathbf{e}_n with entries $(\mathbf{e}_n)_i = \delta_{n,i}$. The dimension of \mathbf{e}_n will be either explicitly stated or will be clear from the context. For N -dimensional unit vectors we use the convention that $\mathbf{e}_n = \mathbf{0}$ for $n \neq 1, \dots, N$.

For the q -binomial (1.1) we need the transformation

$$\begin{bmatrix} L \\ a \end{bmatrix}_{1/q} = q^{-a(L-a)} \begin{bmatrix} L \\ a \end{bmatrix}, \tag{2.2}$$

as well as the recurrences

$$\begin{bmatrix} L \\ a \end{bmatrix} = \begin{bmatrix} L-1 \\ a-1 \end{bmatrix} + q^a \begin{bmatrix} L-1 \\ a \end{bmatrix}, \tag{2.3}$$

$$\begin{bmatrix} L \\ a \end{bmatrix} = \begin{bmatrix} L-1 \\ a \end{bmatrix} + q^{L-a} \begin{bmatrix} L-1 \\ a-1 \end{bmatrix}. \tag{2.4}$$

2.2 Supernomial coefficients

We are now prepared to introduce the supernomial coefficients. Before introducing a base, the $q = 1$ case is briefly considered first. The supernomial coefficients are defined as the following generalization of the multinomial coefficients

Definition 2.1. Let N be a positive integer, $\mathbf{L} \in \mathbb{Z}^N$ and $\boldsymbol{\ell} = T^{-1}\mathbf{L}$. For $a + \frac{1}{2}\ell_N \in \mathbb{Z}_+$ we define the supernomial coefficient $\binom{\mathbf{L}}{a}$ through

$$\prod_{j=1}^N (1 + x + \cdots + x^j)^{L_j} = \sum_{a=-\frac{\ell_N}{2}}^{\infty} \binom{\mathbf{L}}{a} x^{a+\frac{\ell_N}{2}}. \quad (2.5)$$

Clearly, for $\mathbf{L} = L\mathbf{e}_n$ the supernomial simplifies to the $(n-)$ multinomial coefficient (1.3), $\binom{L\mathbf{e}_n}{a} = \binom{L}{a}_n$. Some simple properties of the supernomials are the initial condition and symmetry

$$\binom{\mathbf{0}}{a} = \delta_{a,0} \quad \text{and} \quad \binom{\mathbf{L}}{a} = \binom{\mathbf{L}}{-a} \quad \text{for } \mathbf{L} \in \mathbb{Z}_+^N, \quad (2.6)$$

as well as the recurrences

$$\binom{\mathbf{L}}{a} = \binom{\mathbf{L} + \mathbf{e}_{n-1} - 2\mathbf{e}_n + \mathbf{e}_{n+1}}{a} + \binom{\mathbf{L} - 2\mathbf{e}_n}{a} \quad \text{for } n = 1, \dots, N-1. \quad (2.7)$$

To see this last result we note the identity

$$(1 + x + \cdots + x^n)^2 = (1 + x + \cdots + x^{n-1})(1 + x + \cdots + x^{n+1}) + x^n.$$

A representation useful for defining the q -analogue of the supernomials is provided by

$$\binom{\mathbf{L}}{a} = \sum_{j_1 + \cdots + j_N = a + \frac{\ell_N}{2}} \binom{L_N}{j_N} \binom{L_{N-1} + j_N}{j_{N-1}} \cdots \binom{L_1 + j_2}{j_1}. \quad (2.8)$$

This follows by N -fold use of the binomial expansion,

$$\begin{aligned} \sum_{a=-\frac{\ell_N}{2}}^{\infty} \binom{\mathbf{L}}{a} x^{a+\frac{\ell_N}{2}} &= \sum_{j_1, \dots, j_N \geq 0} x^{j_1 + \cdots + j_N} \prod_{k=1}^N \binom{L_k + j_{k+1}}{j_k} \quad (j_{N+1} = 0) \\ &= (1+x)^{L_1} \sum_{j_2, \dots, j_N} x^{j_2 + \cdots + j_N} (1+x)^{j_2} \prod_{k=2}^N \binom{L_k + j_{k+1}}{j_k} \\ &= (1+x)^{L_1} (1+x+x^2)^{L_2} \sum_{j_3, \dots, j_N} x^{j_3 + \cdots + j_N} (1+x+x^2)^{j_3} \prod_{k=3}^N \binom{L_k + j_{k+1}}{j_k} \\ &= \cdots = \prod_{j=1}^N (1+x+\cdots+x^j)^{L_j}. \end{aligned}$$

2.3 q -Supernomial coefficients

We propose the following q -analogue of (2.8).

Definition 2.2. Let N be a positive integer, $\mathbf{L} \in \mathbb{Z}^N$ and $\boldsymbol{\ell} = T^{-1}\mathbf{L}$. Then for $a + \frac{1}{2}\ell_N \in \mathbb{Z}_+$

$$\left[\begin{matrix} \mathbf{L} \\ a \end{matrix} \right] = \sum_{j_1 + \cdots + j_N = a + \frac{\ell_N}{2}} q^{\sum_{k=2}^N j_{k-1}(L_k + \cdots + L_N - j_k)} \left[\begin{matrix} L_N \\ j_N \end{matrix} \right] \left[\begin{matrix} L_{N-1} + j_N \\ j_{N-1} \end{matrix} \right] \cdots \left[\begin{matrix} L_1 + j_2 \\ j_1 \end{matrix} \right]. \quad (2.9)$$

We also need the $q \rightarrow 1/q$ form of (2.9).

Definition 2.3. *With the same parameters as in definition 2.2 the q -supernomial $T(\mathbf{L}, a)$ is defined as*

$$T(\mathbf{L}, a) = q^{\frac{1}{4}\mathbf{L}T^{-1}\mathbf{L} - \frac{a^2}{N}} \begin{bmatrix} \mathbf{L} \\ a \end{bmatrix}_{1/q}. \quad (2.10)$$

Since $[(L_1, \dots, L_N, 0, \dots, 0)]_a = [(L_1, \dots, L_N)]_a$ this gives

$$T((L_1, \dots, L_N, 0, \dots, 0), a) = q^{\frac{M-N}{MN}a^2} T((L_1, \dots, L_N), a), \quad (2.11)$$

where the vector $(L_1, \dots, L_N, 0, \dots, 0)$ is M -dimensional. Using (2.2) we also deduce that

$$T(\mathbf{L}e_1, a) = q^{\frac{N-1}{N}a^2} \begin{bmatrix} L \\ \frac{L}{2} + a \end{bmatrix}, \quad (2.12)$$

where e_1 is N -dimensional. In section 5 these equations will play quite a remarkable role.

Let us now state an explicit formula for $T(\mathbf{L}, a)$ when $\mathbf{L} \in \mathbb{Z}_+^N$.

Lemma 2.1. *Let N be a positive integer, $\mathbf{L} \in \mathbb{Z}_+^N$ and $\ell = T^{-1}\mathbf{L}$. Set $L_0 = 0$,*

$$x_j = q^{\sum_{k=0}^j (m_k - \frac{L_k}{2})} \quad (j = 1, \dots, N), \quad (2.13)$$

and

$$m_0 = \frac{\ell_1}{2} + \frac{a}{N} - (C^{-1}\mathbf{m})_1, \quad m_N = -\frac{a}{N} - (C^{-1}\mathbf{m})_{N-1}, \quad (2.14)$$

where $\mathbf{m} = (m_1, \dots, m_{N-1})$. Then for $a = -\frac{1}{2}\ell_N, \dots, \frac{1}{2}\ell_N$,

$$T(\mathbf{L}, a) = \sum'_{\substack{m_i \in \mathbb{Z}_+ - \frac{1}{2}L_i \\ 0 \leq i \leq N}} q^{\mathbf{m}C^{-1}\mathbf{m}} \prod_{j=0}^N \frac{(x_j q)_{L_j}}{(q)_{\frac{L_j}{2} + m_j}}, \quad (2.15)$$

where the primed summation symbol denotes a sum over m_0, \dots, m_N , such that (2.14) is satisfied.

In actual use of T it is helpful to note that x_N defined in (2.13) is equal to one. This can be simply observed by noting that $(C^{-1}\mathbf{m})_1 + (C^{-1}\mathbf{m})_{N-1} = m_1 + \dots + m_{N-1}$.

Proof of lemma 2.1. In the following we set $j_{N+1} = 0$. Using (2.2) one may derive

$$\begin{bmatrix} \mathbf{L} \\ a \end{bmatrix}_{1/q} = \sum_{j_1 + \dots + j_N = a + \frac{\ell_N}{2}} q^{\sum_{k=1}^N j_k (j_k - L_k - \dots - L_N)} \prod_{k=1}^N \begin{bmatrix} L_k + j_{k+1} \\ j_k \end{bmatrix}.$$

Rewriting the q -binomials using $(q)_{m+n} = (q)_m (q^{1+m})_n$, we obtain

$$\prod_{k=1}^N \begin{bmatrix} L_k + j_{k+1} \\ j_k \end{bmatrix} = \frac{1}{(q)_{j_1}} \prod_{k=1}^N \frac{(q^{1+j_{k+1}})_{L_k}}{(q)_{L_k + j_{k+1} - j_k}} = \prod_{k=0}^N \frac{(x_k q)_{L_k}}{(q)_{\frac{L_k}{2} + m_k}}.$$

Notice that we used $L_k \geq 0$ to derive the first equality since the rewriting $(q^{L+1})_j/(q)_j = (q^{j+1})_L/(q)_L$ is problematic when $L < 0$. The last step follows from $L_0 = 0$ and the variable change

$$j_k = \sum_{i=0}^{k-1} \left(m_i - \frac{L_i}{2} \right), \quad 1 \leq k \leq N+1, \quad (2.16)$$

where $m_i \in \mathbb{Z} - \frac{L_i}{2}$ ($i = 0, \dots, N$), and m_0 and m_N satisfy (2.14). This variable change makes sense because one can check that $j_{N+1} = 0$, $j_1 + \dots + j_N = a + \frac{\ell_N}{2}$ and $j_k \in \mathbb{Z}$, ($1 \leq k \leq N$). The equality of the exponents

$$\frac{1}{4} \mathbf{L} T^{-1} \mathbf{L} - \frac{a^2}{N} + \sum_{k=1}^N j_k (j_k - L_k - \dots - L_N) = \mathbf{m} C^{-1} \mathbf{m}$$

follows from (2.16) and the explicit form of the inverse Cartan matrices given in (2.1). \square

We note that the q -supernomials for $\mathbf{L} = L\mathbf{e}_n$ ($L \geq 0$) and the q -multinomials of refs. [40, 47] with label $\ell = 0$ coincide,

$$\left[\begin{array}{c} L\mathbf{e}_n \\ a - \frac{nL}{2} \end{array} \right] = \left[\begin{array}{c} L \\ a \end{array} \right]_n^{(0)} \quad \text{and} \quad T(L\mathbf{e}_n, a) = q^{\frac{N-n}{Nn}a^2} T_0^{(n)}(L, a).$$

For general ℓ , the q -multinomials are, however, only equal to the q -supernomials in the limit $L \rightarrow \infty$ (see equation (2.36) below).

2.4 Symmetries and recurrences of the q -supernomial coefficients

In this section we prove the q -analogues of the supernomial symmetry (2.6) and recurrences (2.7). The recurrences play a crucial role in the proof of the polynomial identities in sections 4 and 5.

Lemma 2.2. *For $\mathbf{L} \in \mathbb{Z}_+^N$, the q -supernomials satisfy the symmetries*

$$\left[\begin{array}{c} \mathbf{L} \\ a \end{array} \right] = \left[\begin{array}{c} \mathbf{L} \\ -a \end{array} \right] \quad \text{and} \quad T(\mathbf{L}, a) = T(\mathbf{L}, -a). \quad (2.17)$$

Lemma 2.3. *For $1 \leq n \leq N-1$ and $\mathbf{L} \in \mathbb{Z}^N$, the following recursion relations hold*

$$\left[\begin{array}{c} \mathbf{L} \\ a \end{array} \right] = q^{\ell_n - n} \left[\begin{array}{c} \mathbf{L} - 2\mathbf{e}_n \\ a \end{array} \right] + \left[\begin{array}{c} \mathbf{L} + \mathbf{e}_{n-1} - 2\mathbf{e}_n + \mathbf{e}_{n+1} \\ a \end{array} \right], \quad (2.18)$$

$$T(\mathbf{L}, a) = T(\mathbf{L} - 2\mathbf{e}_n, a) + q^{\frac{1}{2}(L_n - 1)} T(\mathbf{L} + \mathbf{e}_{n-1} - 2\mathbf{e}_n + \mathbf{e}_{n+1}, a). \quad (2.19)$$

The next lemma states which initial conditions are sufficient to determine a function satisfying the q -supernomial recurrences.

Lemma 2.4. *Let X be a function of $\mathbf{L} \in \mathbb{Z}^N$ which obeys the recurrences*

$$X(\mathbf{L}) = q^{\ell_n - n} X(\mathbf{L} - 2\mathbf{e}_n) + X(\mathbf{L} + \mathbf{e}_{n-1} - 2\mathbf{e}_n + \mathbf{e}_{n+1}) \quad (2.20)$$

for $1 \leq n \leq N-1$ and all $\mathbf{L} \in \mathbb{Z}^N$. Then $X(\mathbf{L})$ for $\mathbf{L} \in \mathbb{Z}_+^N$ is uniquely determined by $X(L\mathbf{e}_1)$ with $L \geq 0$.

We now give an analytic proof of lemma 2.3. In the next section, where a partition theoretical interpretation of the q -supernomials will be discussed, we give an alternative, combinatorial proof.

Proof of lemma 2.3. Equation (2.19) follows immediately from (2.18) using (2.10).

For the proof of (2.18) we repeatedly use the q -binomial recurrences (2.3) and (2.4). After inserting (2.9) into the right-hand side of (2.18), we change variables $j_i \rightarrow j_i - 1$ for $1 \leq i \leq n$ in the first term. In the second term we apply (2.4) to the q -binomial containing L_{n+1} . This yields for the right-hand side of (2.18) (with $j_0 = j_{N+1} = 0$),

$$\begin{aligned} & \sum_{j_1 + \dots + j_N = a + \frac{\ell_N}{2}} q^{\sum_{k=2}^N j_{k-1}(L_k + \dots + L_N - j_k)} \left(q^{L_1 + \dots + L_n - j_1 + j_n + j_{n+1} - 1} \prod_{k=1}^N \begin{bmatrix} L_k + j_{k+1} - \delta_{k,n} - \theta(k \leq n) \\ j_k - \theta(k \leq n) \end{bmatrix} \right. \\ & \left. + q^{j_n - j_{n-1}} \prod_{k=1}^N \begin{bmatrix} L_k + j_{k+1} + \delta_{k,n-1} - 2\delta_{k,n} \\ j_k \end{bmatrix} + q^{L_{n+1} - j_{n-1} + j_n - j_{n+1} + j_{n+2} + 1} \prod_{k=1}^N \begin{bmatrix} L_k + j_{k+1} + \delta_{k,n-1} - 2\delta_{k,n} \\ j_k - \delta_{k,n+1} \end{bmatrix} \right), \end{aligned}$$

where $\theta(\text{true}) = 1$ and $\theta(\text{false}) = 0$. Inserting the telescopic expansion

$$\prod_{k=1}^{n-1} \begin{bmatrix} L_k + j_{k+1} + \delta_{k,n-1} \\ j_k \end{bmatrix} = \sum_{m=0}^{n-1} q^{jm} \prod_{k=1}^{n-1} \begin{bmatrix} L_k + j_{k+1} + \delta_{k,n-1} - \theta(m \leq k < n) \\ j_k - \theta(m < k < n) \end{bmatrix}$$

in the second term, yields

$$\begin{aligned} & \sum_{j_1 + \dots + j_N = a + \frac{\ell_N}{2}} q^{\sum_{k=2}^N j_{k-1}(L_k + \dots + L_N - j_k)} \left(q^{L_{n+1} - j_{n-1} + j_n - j_{n+1} + j_{n+2} + 1} \prod_{k=1}^N \begin{bmatrix} L_k + j_{k+1} + \delta_{k,n-1} - 2\delta_{k,n} \\ j_k - \delta_{k,n+1} \end{bmatrix} \right. \\ & \quad \left. + \sum_{m=0}^{n-1} q^{jm - j_{n-1} + j_n} \prod_{k=1}^N \begin{bmatrix} L_k + j_{k+1} + \delta_{k,n-1} - 2\delta_{k,n} - \theta(m \leq k < n) \\ j_k - \theta(m < k < n) \end{bmatrix} \right. \\ & \quad \left. + q^{L_1 + \dots + L_n - j_1 + j_n + j_{n+1} - 1} \prod_{k=1}^N \begin{bmatrix} L_k + j_{k+1} - \delta_{k,n} - \theta(k \leq n) \\ j_k - \theta(k \leq n) \end{bmatrix} \right). \quad (2.21) \end{aligned}$$

Now change $j_{n+1} \rightarrow j_{n+1} + 1$ and $j_n \rightarrow j_n - 1$ in the first term and $j_n \rightarrow j_n - 1$ and $j_{m+1} \rightarrow j_{m+1} + 1$ in the m -th term in the sum over m . This leads to (changing $m \rightarrow m - 1$ and including the last term in (2.21) as the $m = 0$ term in the sum)

$$\begin{aligned} & \sum_{j_1 + \dots + j_N = a + \frac{\ell_N}{2}} q^{\sum_{k=2}^N j_{k-1}(L_k + \dots + L_N - j_k)} \left(\prod_{k=1}^N \begin{bmatrix} L_k + j_{k+1} - \delta_{k,n} \\ j_k - \delta_{k,n} \end{bmatrix} \right. \\ & \quad \left. + \sum_{m=0}^n q^{L_{m+1} + \dots + L_n - j_{m+1} + j_n + j_{n+1} + \delta_{m,n} - 1} \prod_{k=1}^N \begin{bmatrix} L_k + j_{k+1} - \delta_{k,n} - \theta(m \leq k \leq n) \\ j_k - \theta(m < k \leq n) \end{bmatrix} \right). \end{aligned}$$

Now one can telescopically combine all the terms, starting with combining the $m = 0$ and $m = 1$ terms in the sum using (2.4), then combining this with the $m = 2$ term, and so on. The result of this can be combined with the term in the first line using (2.3). This yields (2.9) and we are done. \square

Proof of lemma 2.4. We will show that the recurrences

$$\begin{aligned} X(\mathbf{L} + \mathbf{e}_{n+1}) &= q^{-\ell_{n-2}-2(n-2)} \left(X(\mathbf{L} + \mathbf{e}_{n-2} + \mathbf{e}_{n-1} + \mathbf{e}_n) - q^{\ell_{n-1}+2n-3} X(\mathbf{L} + \mathbf{e}_{n-1}) \right. \\ &\quad \left. - \theta(n > 2) (q^{\ell_n+2n-3} X(\mathbf{L} + \mathbf{e}_{n-3}) + X(\mathbf{L} + \mathbf{e}_{n-3} + \mathbf{e}_{n-1} + \mathbf{e}_{n+1})) \right) \end{aligned} \quad (2.22)$$

for $2 \leq n \leq N-1$ and $L_{n-1} = 0$ follow from the recurrences (2.20). These recurrences together with the recurrences (2.20), both for $\mathbf{L} \in \mathbb{Z}_+^N$, uniquely determine $X(\mathbf{L})$ for $\mathbf{L} \in \mathbb{Z}_+^N$ from the initial condition $X(L\mathbf{e}_1)$ ($L \geq 0$) for the following reason. First, $X(L_1\mathbf{e}_1 + L_2\mathbf{e}_2)$ for $L_1, L_2 \geq 0$ is determined by recurrence (2.20) for $n = 1$. Now assume that $X(\mathbf{L}')$ is known for all $\mathbf{L}' = L'_1\mathbf{e}_1 + \cdots + L'_{n+1}\mathbf{e}_{n+1}$ ($n \geq 1$) with $L'_i \in \mathbb{Z}_+$ ($1 \leq i \leq n$) and $0 \leq L'_{n+1} < L_{n+1}$ for some $L_{n+1} > 1$. It then suffices to show that $X(\mathbf{L})$ with $\mathbf{L} = L_1\mathbf{e}_1 + \cdots + L_{n+1}\mathbf{e}_{n+1}$ and $L_i \in \mathbb{Z}_+$ ($1 \leq i \leq n$) is determined by the recurrences (2.20) and (2.22). Rewriting (2.20) as

$$X(\mathbf{L} + \mathbf{e}_{n+1}) = X(\mathbf{L} - \mathbf{e}_{n-1} + 2\mathbf{e}_n) - q^{\ell_n+1} X(\mathbf{L} - \mathbf{e}_{n-1}) \quad (2.23)$$

we see that $X(\mathbf{L})$ with $L_{n-1} > 0$ is determined from $X(\mathbf{L}')$ with only positive components L'_i . $X(\mathbf{L})$ with $L_{n-1} = 0$ now follows from (2.22) since all terms on the right-hand side are known and have positive components.

It remains to be shown that (2.22) can indeed be deduced from (2.20) for $\mathbf{L} \in \mathbb{Z}^N$. We start with (2.23) and apply (2.20) with n replaced by $n-1$ to the first term on the right-hand side

$$\begin{aligned} X(\mathbf{L} + \mathbf{e}_{n+1}) &= X(\mathbf{L} - \mathbf{e}_{n-2} + \mathbf{e}_{n-1} + \mathbf{e}_n) - q^{\ell_{n-1}+1} X(\mathbf{L} - \mathbf{e}_{n-2} - \mathbf{e}_{n-1} + \mathbf{e}_n) \\ &\quad - q^{\ell_n+1} X(\mathbf{L} - \mathbf{e}_{n-1}). \end{aligned}$$

Once more applying (2.20) with $n \rightarrow n-1$ now to the second term on the right-hand side yields

$$\begin{aligned} X(\mathbf{L} + \mathbf{e}_{n+1}) &= X(\mathbf{L} - \mathbf{e}_{n-2} + \mathbf{e}_{n-1} + \mathbf{e}_n) - q^{\ell_{n-1}+1} X(\mathbf{L} - 2\mathbf{e}_{n-2} + \mathbf{e}_{n-1}) \\ &\quad + q^{2\ell_{n-1}-2n+5} X(\mathbf{L} - 2\mathbf{e}_{n-2} - \mathbf{e}_{n-1}) - q^{\ell_n+1} X(\mathbf{L} - \mathbf{e}_{n-1}). \end{aligned} \quad (2.24)$$

Observe that for $n = 2$ and $L_1 = 0$ the last two terms cancel yielding (2.22) for $n = 2$. For $n > 2$, the last two terms combine to $-q^{\ell_n+1} X(\mathbf{L} + \mathbf{e}_{n-3} - 2\mathbf{e}_{n-2})$ by (2.20) if $L_{n-1} = 0$. Replacing $X(\mathbf{L} + \mathbf{e}_{n+1})$ on the left-hand side of (2.24) by

$$X(\mathbf{L} + \mathbf{e}_{n+1}) = q^{\ell_{n-2}} X(\mathbf{L} - 2\mathbf{e}_{n-2} + \mathbf{e}_{n+1}) + X(\mathbf{L} + \mathbf{e}_{n-3} - 2\mathbf{e}_{n-2} + \mathbf{e}_{n-1} + \mathbf{e}_{n+1}),$$

solving for $X(\mathbf{L} - 2\mathbf{e}_{n-2} + \mathbf{e}_{n+1})$ and replacing $\mathbf{L} \rightarrow \mathbf{L} + 2\mathbf{e}_{n-2}$ yields (2.22). \square

Proof of lemma 2.2. Lemma 2.2 follows from lemmas 2.3 and 2.4 by induction. Setting $X(\mathbf{L}) = \begin{bmatrix} \mathbf{L} \\ a \end{bmatrix}$ and recalling that the q -supernomial fulfills the recurrence (2.18) we know from lemma 2.4 and its proof that $\begin{bmatrix} \mathbf{L} \\ a \end{bmatrix}$ for $\mathbf{L} \in \mathbb{Z}_+^N$ is determined from $\begin{bmatrix} L\mathbf{e}_1 \\ a \end{bmatrix}$ ($L \geq 0$) from recurrences only involving $\begin{bmatrix} \mathbf{L} \\ a \end{bmatrix}$ with $\mathbf{L} \in \mathbb{Z}_+^N$. Since $\begin{bmatrix} L\mathbf{e}_1 \\ a \end{bmatrix} = \begin{bmatrix} L\mathbf{e}_1 \\ -a \end{bmatrix}$ thanks to the symmetry of the q -binomial the lemma is proven. \square

2.5 Limiting behaviour of the q -supernomials

By definition both types of q -supernomials are polynomials in q . In the limit $L_m \rightarrow \infty$ the q -supernomials yield q -series, as described in the next few lemmas.

Lemma 2.5. *For $m = 1, 2, \dots, N$ and $|q| < 1$,*

$$\lim_{L_m \rightarrow \infty} \left[a - \frac{\ell_N}{2} \right]_{\mathbf{L}} = \frac{1}{(q)_a}, \quad (2.25)$$

independent of m .

Proof. First notice that for $|q| < 1$

$$q^{\sum_{k=2}^m j_{k-1}(L_k + \dots + L_N - j_k)}$$

is only non-zero for $L_m \rightarrow \infty$ if $j_1 = \dots = j_{m-1} = 0$. This implies

$$\lim_{L_m \rightarrow \infty} \left[a - \frac{\ell_N}{2} \right]_{\mathbf{L}} = \sum_{j_m + \dots + j_N = a} q^{\sum_{k=m+1}^N j_{k-1}(L_k + \dots + L_N - j_k)} \left[\begin{matrix} L_N \\ j_N \end{matrix} \right] \dots \left[\begin{matrix} L_{m+1} + j_{m+2} \\ j_{m+1} \end{matrix} \right] \frac{1}{(q)_{j_m}}.$$

Hence it is sufficient to show that

$$\lim_{L_1 \rightarrow \infty} \left[a - \frac{\ell_N}{2} \right]_{\mathbf{L}} = \frac{1}{(q)_a}.$$

Equation (2.18) implies that

$$\lim_{L_1 \rightarrow \infty} \left[a - \frac{\ell_N}{2} \right]_{\mathbf{L}} = \lim_{L_1 \rightarrow \infty} \left[\begin{matrix} \mathbf{L} + \mathbf{e}_{n-1} - 2\mathbf{e}_n + \mathbf{e}_{n+1} \\ a - \frac{\ell_N}{2} \end{matrix} \right] \quad (2.26)$$

for $n = 1, 2, \dots, N-1$. From (2.26) with $n = 1$ it follows that $\lim_{L_1 \rightarrow \infty} \left[a - \frac{\ell_N}{2} \right]_{\mathbf{L}}$ is independent of L_2 . From (2.26) with $n = 2$ one deduces that it is independent of L_3 , et cetera. We can therefore choose $L_2 = \dots = L_N = 0$, which immediately implies (2.25). \square

In contrast to the result of lemma 2.5, the various limits of $T(\mathbf{L}, a)$ depend on which components of \mathbf{L} are taken to infinity, and to give our results we need some definitions first.

Definition 2.4. *Let N, h be integers such that $1 \leq h \leq N$. Choose integers k_i ($1 \leq i \leq h$) such that $1 \leq k_1 < k_2 < \dots < k_h \leq N$ and denote the sets $K = \{k_1, k_2, \dots, k_h\}$ and $\bar{K} = \{1, 2, \dots, N\} - K$. Finally, fix $a \in \mathbb{Z}$, $L_k \in \mathbb{Z}_+$ for $k \in \bar{K}$ and $\sigma_k = 0, 1$ for $k \in K$ such that $a + \sum_{k \in \bar{K}} kL_k + \sum_{k \in K} k\sigma_k$ is even. Then*

$$b_a^{\{L_k | k \in \bar{K}\} \{\sigma_k | k \in K\}}(q) = \sum_{\substack{m_i \in \mathbb{Z}_+ - \frac{1}{2}L_i, \ i \in \bar{K} \\ m_i \in \mathbb{Z} - \frac{1}{2}\sigma_i, \ i \in K \\ \frac{a}{2N} + (C^{-1}\mathbf{m})_{N-1} = -m_N}} q^{mC^{-1}\mathbf{m}} \frac{(x_{k_h}q)_\infty}{(q)_\infty^{h+1}} \frac{\prod_{j=k_h+1}^N (x_jq)^{L_j}}{\prod_{j \in \bar{K}} (q)^{\frac{L_j}{2} + m_j}}, \quad (2.27)$$

with the variables x_j given by

$$x_j = q^{\frac{a}{2N} + \frac{1}{2}(L_{j+1} + \dots + L_N) + (\mathbf{e}_j - \mathbf{e}_{j+1})C^{-1}\mathbf{m}} \quad (k_h \leq j < N) \quad \text{and} \quad x_N = 1. \quad (2.28)$$

Definition 2.5. Let N be a positive integer, $\mathbf{L} \in \mathbb{Z}_+^{N-1}$, $a \in \mathbb{Z}$ and $\sigma = 0, 1$ such that $r = a - N(C^{-1}\mathbf{L})_{N-1} + N\sigma$ is even. Then

$$c_a^{\mathbf{L}, \sigma}(q) = \frac{q^{\frac{\mathbf{L}C^{-1}\mathbf{L}}{2(N+2)}}}{(q)_\infty} \sum_{\substack{\mathbf{m} \in \mathbb{Z}_+^{N-1} \\ \frac{r}{2N} - (C^{-1}\mathbf{m})_1 \in \mathbb{Z}}} \frac{q^{\mathbf{m}C^{-1}(\mathbf{m}-\mathbf{L})}}{(q)_{m_1} \cdots (q)_{m_{N-1}}}. \quad (2.29)$$

The function $c_a^{\mathbf{L}, \sigma}(q)$ has the symmetries

$$c_a^{\mathbf{L}, \sigma} = c_{-a}^{\mathbf{L}, \sigma} = c_{a+2N}^{\mathbf{L}, \sigma} = c_{N-a}^{\mathbf{L}, 1-\sigma} = c_a^{\mathbf{L}', \sigma'}, \quad (2.30)$$

with $L'_j = L_{N-j}$ and $\sigma' \equiv \sigma + L_1 + \cdots + L_{N-1} \pmod{2}$. For $\mathbf{L} = \mathbf{e}_\ell$ ($0 \leq \ell < N$) it reduces to the level- N $A_1^{(1)}$ string functions c_a^ℓ [29, 30],

$$c_a^\ell = c_{a+N\sigma}^{\mathbf{e}_\ell, \sigma}. \quad (2.31)$$

Lemma 2.6. Let N be a positive integer, $\mathbf{L} \in \mathbb{Z}_+^N$ and $|q| < 1$. Define a, h, k_i ($1 \leq i \leq h$), σ_{k_i} and K, \bar{K} as in definition 2.4. Then

$$b_a^{\{L_k | k \in \bar{K}\} \{\sigma_k | k \in K\}}(q) = \lim_{\substack{L_{k_1}, \dots, L_{k_h} \rightarrow \infty \\ L_{k_i} \equiv \sigma_{k_i} \pmod{2}, (1 \leq i \leq h)}} T\left(\mathbf{L}, \frac{a}{2}\right). \quad (2.32)$$

Proof. This follows immediately from lemma 2.1 noting that x_j in (2.13) may be rewritten (by inserting the explicit form of m_0 and replacing $a \rightarrow a/2$) as (2.28) for all $1 \leq j \leq N$. This implies for $|q| < 1$,

$$\lim_{L_k \rightarrow \infty} x_j = \begin{cases} 0 & \text{for } j < k \\ x_j & \text{for } j \geq k. \end{cases} \quad \square$$

Recalling that $x_N = 1$, equation (2.27) with $k_h = N$ simplifies to

$$b_a^{\{L_k | k \in \bar{K}\} \{\sigma_k | k \in K\}}(q) = \frac{1}{(q)_\infty^h} \sum_{\substack{m_i \in \mathbb{Z} + \frac{1}{2}L_i, i \in \bar{K} \\ m_i \in \mathbb{Z} - \frac{1}{2}\sigma_i, i \in K - \{N\} \\ \frac{a}{2N} + (C^{-1}\mathbf{m})_{N-1} \in \mathbb{Z} + \frac{1}{2}\sigma_N}} q^{\mathbf{m}C^{-1}\mathbf{m}} \prod_{j \in \bar{K}} \frac{1}{(q)_{\frac{L_j}{2} + m_j}}. \quad (2.33)$$

When further $h = 1$ we may shift $m_j \rightarrow m_j - L_j/2$ ($1 \leq j < N$) and obtain for $\mathbf{L} \in \mathbb{Z}_+^{N-1}$

$$b_a^{\{L_k\} \{\sigma\}}(q) = q^{\frac{N}{4(N+2)}\mathbf{L}C^{-1}\mathbf{L}} c_a^{\mathbf{L}, \sigma}(q). \quad (2.34)$$

This leads to

Corollary 2.1. Let N be a positive integer, $\mathbf{L} \in \mathbb{Z}_+^{N-1}$, $a \in \mathbb{Z}$ and $\sigma = 0, 1$, such that $a + N(C^{-1}\mathbf{L})_{N-1} + N\sigma$ is even. Then

$$c_a^{\mathbf{L}, \sigma}(q) = q^{-\frac{N}{4(N+2)}\mathbf{L}C^{-1}\mathbf{L}} \lim_{\substack{L_N \rightarrow \infty \\ L_N \equiv \sigma \pmod{2}}} T\left((\mathbf{L}, L_N), \frac{a}{2}\right). \quad (2.35)$$

Finally, for $\mathbf{L} = \mathbf{e}_\ell$ ($1 \leq \ell \leq N$), the limit of the q -supernomials is related to the limit of the q -multinomials $T_\ell^{(N)}(L, a)$ of ref. [40],

$$\lim_{\substack{L \rightarrow \infty \\ L \equiv \sigma \pmod{2}}} T(L\mathbf{e}_N + \mathbf{e}_\ell, a) = q^{\frac{\ell(N-\ell)}{4N}} \lim_{\substack{L \rightarrow \infty \\ L \equiv \sigma \pmod{2}}} T_\ell^{(N)}\left(L, \frac{\ell}{2} - a\right). \quad (2.36)$$

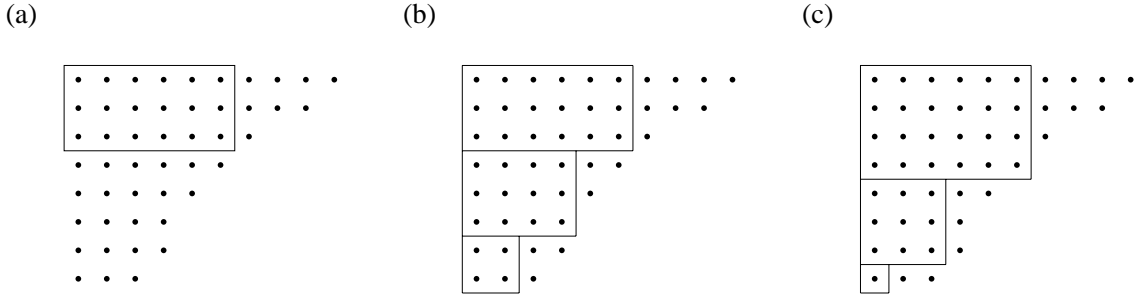


Figure 1: (a) Durfee rectangle of excess 3 of the partition π . (b) The $(3, 1, 0)$ -Durfee dissection of π . (c) The $(2, 0, 0)$ -Durfee dissection of π .

3 q -Supernomial coefficients and partitions

In the following a partition theoretic interpretation of the q -supernomials is provided. As we will see, this interpretation can be used to prove the recurrences of lemma 2.3 and the limits of lemma 2.5 combinatorially. Before delving into the combinatorics of supernomials, let us introduce a slight modification of the q -supernomial $T(\mathbf{L}, a)$, (compare with (2.10)),

$$\tilde{T}(\mathbf{L}, a) = q^{a\ell_1} \left[a - \frac{\ell_N}{2} \right]_{1/q}^{\mathbf{L}}. \quad (3.1)$$

The explicit form for \tilde{T} is easily found to be

$$\tilde{T}(\mathbf{L}, a) = \sum_{j_1 + \dots + j_N = a} q^{\sum_{k=1}^N j_k(j_k + L_1 + \dots + L_{k-1})} \begin{bmatrix} L_N \\ j_N \end{bmatrix} \begin{bmatrix} L_{N-1} + j_N \\ j_{N-1} \end{bmatrix} \dots \begin{bmatrix} L_1 + j_2 \\ j_1 \end{bmatrix}. \quad (3.2)$$

3.1 q -Supernomials and Durfee dissections

In order to interpret equation (3.2) we first give some basic definitions about Durfee dissections, generalizing some of the concepts introduced in refs. [6, 47]. In the following a partition and its Ferrers graph are identified.

Definition 3.1. *The Durfee rectangle of excess E of a partition is the maximal rectangle of nodes with E more columns than rows.*

As an example, the Durfee rectangle of excess 3 of the partition $\pi = 10 + 9 + 7 + 6 + 5 + 4 + 4 + 3$ is drawn in figure 1(a).

For brevity it is often convenient to suppress the phrase “of excess E ” and in the following we sometimes refer to just “the Durfee rectangle”, assuming that its excess has been fixed.

Definition 3.2. *The width of a Durfee rectangle is the number of columns. The height of a Durfee rectangle is the number of rows.*

The part of a partition below its Durfee rectangle is again a partition (with parts less than or equal to the width of the Durfee rectangle) and we can define a second Durfee rectangle, then a third and so on.

Definition 3.3. The (E_1, \dots, E_n) -Durfee dissection of a partition is obtained by successively drawing Durfee rectangles of excess E_1, E_2, \dots, E_n .

The $(3, 1, 0)$ -Durfee dissection of π is shown in figure 1(b).

As a final definition we have

Definition 3.4. Let N be a positive integer and set $\mathbf{L} \in \mathbb{Z}_+^N$. A partition is $(\mathbf{L}; a)$ -admissible if it has

1. exactly a parts,
2. no parts exceeding $L_1 + L_2 + \dots + L_N$,
3. no parts below its $(L_{N-1} + \dots + L_1, \dots, L_2 + L_1, L_1, 0)$ -Durfee dissection.

The set of $(\mathbf{L}; a)$ -admissible partitions is denoted as $S_{\mathbf{L}; a}$.

Using 1, the third condition is of course equivalent to the condition that the heights of the successive Durfee rectangles add up to a .

For $L \geq 7$, the partition π is $(1, 2, L; 8)$ -admissible as well as $(0, 2, L + 1; 8)$ -admissible, see figure 1(b) and (c), respectively. Note that the set $S_{L\mathbf{e}_N; a}$ corresponds to the set of partitions with at most N successive Durfee squares, such that the number of parts is a and no part exceeds L .

We are now ready for the first claim of this section.

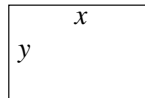
Lemma 3.1. For $\mathbf{L} \in \mathbb{Z}_+^N$, $\tilde{T}(\mathbf{L}, a)$ is the generating function of $(\mathbf{L}; a)$ -admissible partitions.

Proof. Let λ be an $(\mathbf{L}; a)$ -admissible partition, and let j_ℓ be the height of the ℓ -th successive Durfee rectangle D_ℓ counted from below ($\ell = 1, \dots, N$). The number of nodes of D_ℓ is $j_\ell(j_\ell + L_1 + \dots + L_{\ell-1})$. The part of λ to the right of D_ℓ (and below $D_{\ell+1}$ when $\ell < N$) is a partition with largest part $\leq j_{\ell+1} - j_\ell + L_\ell$ (with $j_{N+1} = 0$) and number of parts $\leq j_\ell$. Since the generating function of such partitions is given by $\left[\begin{smallmatrix} L_\ell + j_{\ell+1} \\ j_\ell \end{smallmatrix} \right]$ we find that the generating function of $(\mathbf{L}; a)$ -admissible partitions reads

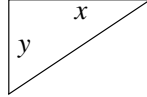
$$\sum_{j_1 + \dots + j_N = a} \prod_{\ell=1}^N q^{j_\ell(j_\ell + L_1 + \dots + L_{\ell-1})} \left[\begin{smallmatrix} L_\ell + j_{\ell+1} \\ j_\ell \end{smallmatrix} \right] = \tilde{T}(\mathbf{L}, a).$$

Here the restriction in the sum over j_ℓ arises from the fact that the sum over the respective heights of the Durfee rectangles should add up to a , to ensure that condition 3 of definition 3.4 is satisfied. \square

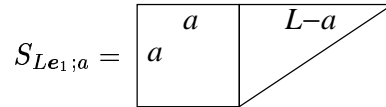
Having established the above result it is convenient to introduce a graphical notation for the set $S_{\mathbf{L}; a}$. We proceed to do so in several steps. First, a rectangle



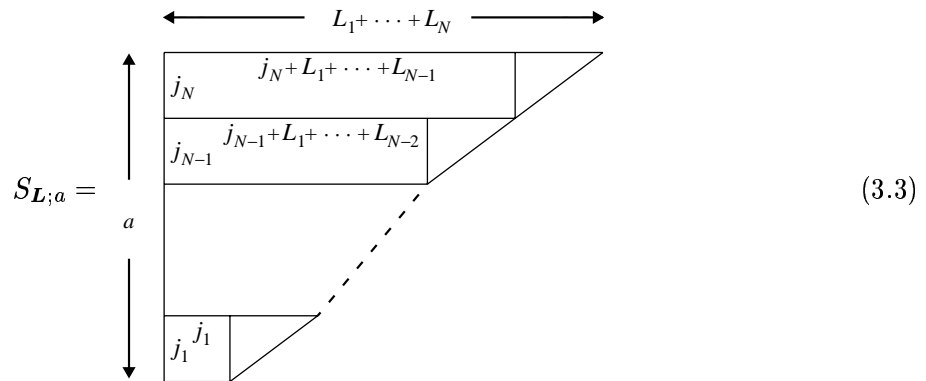
represents the partition of xy consisting of y rows of x nodes. Alternatively, we can view the above rectangle as the (trivial) set of partitions with y parts, all parts having size x . Second, the triangle



represents the set of partitions with at most y parts, no part exceeding x . We now combine these two elements to represent more complicated sets of partitions such as $S_{\mathbf{L};a}$. As an example we first take $\mathbf{L} = L\mathbf{e}_1$, in which case $S_{\mathbf{L};a}$ corresponds to the set of partitions with exactly a parts, no part being less than a and no part exceeding L . Graphically this becomes



We assume that through these examples the idea is clear, and for general \mathbf{L} we obtain the following graphical representation of $S_{\mathbf{L};a}$:



To reiterate, to find a partition (or its Ferrers graph) in the set $S_{\mathbf{L};a}$ one interprets the above graph as follows. Choose integers j_1, \dots, j_N such that their sum is a and draw N successive rectangles of j_k times $j_k + L_1 + \dots + L_{k-1}$ nodes. Then, to the right of the k -th rectangle from below, one may draw the Ferrers graph of any partition that has no more than j_k parts and that has largest part not exceeding $L_k + j_{k+1} - j_k$ (with $j_{N+1} = 0$).

In the next section we make frequent use of (parts of) this graph. When we refer to “the n -th rectangle (triangle)” this is meant to indicate the n -th rectangle (triangle) counted from below. Also, to allow for a more compact notation, we often omit (part of) the labels assuming that the excess of the n -th rectangle is $L_1 + \dots + L_{n-1}$ unless stated otherwise.

3.2 Combinatorial proof of lemma 2.3

The previously introduced graphical notation will aid us in asserting lemma 2.3 purely combinatorially for $\mathbf{L} - 2\mathbf{e}_n \in \mathbb{Z}_+^N$. In terms of $\tilde{T}(\mathbf{L}, a)$ the recurrences (2.19) read

$$\tilde{T}(\mathbf{L}, a) = \tilde{T}(\mathbf{L} + \mathbf{e}_{n-1} - 2\mathbf{e}_n + \mathbf{e}_{n+1}, a) + q^{2a-n+\sum_{i=1}^{n-1} (n-i)L_i} \tilde{T}(\mathbf{L} - 2\mathbf{e}_n, a - n), \quad (3.4)$$

for $n = 2, \dots, N - 1$, and

$$\tilde{T}(\mathbf{L}, a) = q^a \tilde{T}(\mathbf{L} - 2\mathbf{e}_1 + \mathbf{e}_2, a) + q^{2a-1} \tilde{T}(\mathbf{L} - 2\mathbf{e}_1, a - 1). \quad (3.5)$$

3.2.1 Recurrences (3.4)

Inspection of equation (3.4) shows that on the left-hand side we have the generating function of $(\mathbf{L}; a)$ -admissible partitions, whereas on the right-hand side we have the generating function of $(\mathbf{L} + \mathbf{e}_{n-1} - 2\mathbf{e}_n + \mathbf{e}_{n+1}; a)$ -admissible partitions plus the generating function of $(\mathbf{L} - 2\mathbf{e}_n; a - n)$ -admissible partitions multiplied by q to the power $2a - n + \sum_{i=1}^{n-1} (n - i)L_i$. We now show that $S_{\mathbf{L} + \mathbf{e}_{n-1} - 2\mathbf{e}_n + \mathbf{e}_{n+1}; a} \subseteq S_{\mathbf{L}; a}$ and, setting $T_{\mathbf{L}; a} = S_{\mathbf{L}; a} - S_{\mathbf{L} + \mathbf{e}_{n-1} - 2\mathbf{e}_n + \mathbf{e}_{n+1}; a}$, that there is a bijection between $T_{\mathbf{L}; a}$ and $S_{\mathbf{L} - 2\mathbf{e}_n; a - n}$, such that we have to remove $2a - n + \sum_{i=1}^{n-1} (n - i)L_i$ nodes from each element of $T_{\mathbf{L}; a}$ to map it to an element of $S_{\mathbf{L} - 2\mathbf{e}_n; a - n}$.

To show that $S_{\mathbf{L} + \mathbf{e}_{n-1} - 2\mathbf{e}_n + \mathbf{e}_{n+1}; a} \subseteq S_{\mathbf{L}; a}$ we make repeated use of the identities

$$\begin{array}{c} x \\ \diagdown \\ y \end{array} = \begin{array}{c} x \\ \diagdown \\ y \end{array} \dot{\cup} \begin{array}{c} x-1 \\ \diagdown \\ y \end{array} \quad (3.6)$$

and

$$\begin{array}{c} x \\ \diagdown \\ y \end{array} = \begin{array}{c} x \\ \diagdown \\ y \\ \hline 1 \end{array} \dot{\cup} \begin{array}{c} x \\ \diagdown \\ y-1 \end{array} \quad (3.7)$$

where it will be customary not to explicitly draw the rectangles of size $1 \times y$ and $x \times 1$. The symbol $\dot{\cup}$ stands for the disjoint union. Of course, (3.6) just reflects the fact that a partition with at most y parts and no part exceeding x is either a partition with largest part precisely x , or a partition with no parts exceeding $x - 1$ (with in both cases no more than y parts), and translated into generating functions (3.6) corresponds to (2.3) with $L = x + y$ and $a = x$. Similarly, (3.7) encodes that a partition with at most y parts and no part exceeding x is either a partition with precisely y parts or a partition with at most $y - 1$ parts (with in both cases no part exceeding x), corresponding to (2.4) with $L = x + y$ and $a = x$.

Using (3.6) we may write

$$(3.3) = \begin{array}{c} j_{n+1} \\ \hline j_n \end{array} \begin{array}{c} \diagdown \\ \diagdown \\ \diagdown \end{array} \dot{\cup} \begin{array}{c} j_{n+1} \\ \hline j_n \end{array} \begin{array}{c} \diagdown \\ \diagdown \\ \diagdown \end{array}$$

We now deform the first graph by increasing the height of the $(n + 1)$ -th rectangle by one thereby decreasing the height of n -th rectangle. Since this only changes the interior of the graph, it still represents the same set of partitions. After this change, we shift variables $j_{n+1} \rightarrow j_{n+1} - 1$ and $j_n \rightarrow j_n + 1$, so that j_n and j_{n+1} again label the heights of the n -th and $(n + 1)$ -th rectangle. The second graph is rewritten using (3.7) on the n -th triangle. In all this gives that (3.3) equals

$$\begin{array}{c} n+1 \\ \hline n \end{array} \begin{array}{c} -1 \\ \diagdown \\ \diagdown \end{array} \dot{\cup} \begin{array}{c} n+1 \\ \hline n \end{array} \begin{array}{c} \diagdown \\ \diagdown \\ \diagdown \end{array} \dot{\cup} \begin{array}{c} n+1 \\ \hline n \end{array} \begin{array}{c} \diagdown \\ \diagdown \\ \diagdown \end{array}$$

all this is the graph

(3.8)

which represents precisely the set of $(\mathbf{L} + e_{n-1} - 2e_n + e_{n+1}; a)$ -admissible partitions.

Let us now consider the neglected $(n + 2)$ -th graph representing the set $T_{\mathbf{L};a} = S_{\mathbf{L};a} - S_{\mathbf{L}+e_{n-1}-2e_n+e_{n+1};a}$. The non-trivial part of this graph reads

(3.9)

We wish to show that there is a bijection between the set represented by this graph and the set $S_{\mathbf{L}-2e_n;a-n}$, such that each element of $T_{\mathbf{L};a}$ is mapped to an element of $S_{\mathbf{L}-2e_n;a-n}$ by removal of $2a - n + \sum_{i=1}^{n-1} (n-i)L_i$ nodes. In fact, it turns out that the map from a partition in $T_{\mathbf{L};a}$ to a partition in $S_{\mathbf{L}-2e_n;a-n}$ acts only on the Durfee rectangles, and is independent of the particular details of the partition. We therefore temporarily change our interpretation of the graphs, and view the graph in equation (3.9) as not representing the set $T_{\mathbf{L};a}$, but as representing an arbitrary element of $T_{\mathbf{L};a}$. That is, at the position of the k -th triangle we assume an arbitrary partition with number of parts less than or equal to the height of the triangle and largest part less than or equal to the width of the triangle. Then we remove the last row of the the first n rectangles, the first column and the column just preceding the $(n + 1)$ -th triangle. Marking these rows and columns in grey, we get

Since we have removed n rows and two columns the right-hand side has $a - n$ rows and no more than $L_1 + \dots + L_N - 2$ columns, and we conclude that it represents the set $S_{\mathbf{L}-2e_n;a-n}$ (or arbitrary partitions in that set, depending which interpretation of the graph is chosen). Since our map is trivially reversible, we are done.

3.2.2 Recurrence (3.5)

The proof of the exceptional case (3.5) proceeds in identical fashion to the proof of (3.4). The only modification arises from the fact that the set S , represented by the graph of equation (3.8) with $n = 1$ (so that the dotted lines below the graph should be omitted), does not represent the set $S_{\mathbf{L}-2e_1+e_2;a}$. However, the map onto $S_{\mathbf{L}-2e_1+e_2;a}$ is trivial, since we only have to remove the first column of (3.8). Since this corresponds to removal of a nodes from each element of S this gives rise to the additional factor q^a in the first term of the right-hand side of (3.5). Obviously this map is reversible, completing the proof.

3.2.3 Recurrence for $n = N$

The recurrences in lemma 2.3 hold for $n = 1, \dots, N - 1$. Using the above graphical methods, it is an easy matter to verify that on setting $n = N$ one gets the modified result

$$\begin{aligned} \tilde{T}(\mathbf{L}, a) &= q^{\ell_1} \tilde{T}(\mathbf{L} + \mathbf{e}_{N-1} - \mathbf{e}_N, a - 1) + q^{2a-N+\sum_{i=1}^N (N-i)L_i} \tilde{T}(\mathbf{L} - 2\mathbf{e}_N, a - N) \\ &\quad + \tilde{T}(\mathbf{L} + \mathbf{e}_{N-1} - 2\mathbf{e}_N, a). \end{aligned} \quad (3.10)$$

Similarly one can find that

$$\begin{aligned} \tilde{T}(\mathbf{L}, a) &= \tilde{T}(\mathbf{L} + \mathbf{e}_{N-1} - \mathbf{e}_N, a) + q^{2a-N+\sum_{i=1}^N (N-i)L_i} \tilde{T}(\mathbf{L} - 2\mathbf{e}_N, a - N) \\ &\quad + q^{2a-N-1+\sum_{i=1}^N (N-i+1)L_i} \tilde{T}(\mathbf{L} + \mathbf{e}_{N-1} - 2\mathbf{e}_N, a - N - 1). \end{aligned} \quad (3.11)$$

Both these results, which hold for $N \geq 2$ and $\mathbf{L} - 2\mathbf{e}_N \in \mathbb{Z}_+^N$, will be briefly touched upon in section 4.2. However, contrary to the cases $1 \leq n \leq N - 1$, neither will be essential in any of the proofs of subsequent theorems.

3.3 A Durfee rectangle like identity

A last combinatorial aspect of the q -supernomials we wish to mention is that one can prove lemma 2.5 by interpreting equation (2.25) as a successive Durfee rectangle identity.

Let us first recall the Durfee rectangle identity

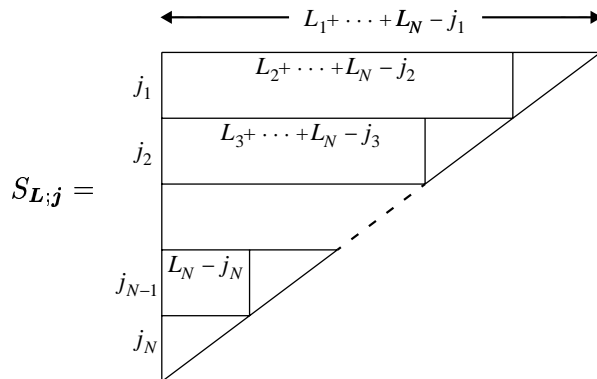
$$\sum_{n=m}^{\infty} \frac{q^{(n-m)(n+m)}}{(q)_{n-m}(q)_{n+m}} = \frac{1}{(q)_{\infty}},$$

true for all (integer) $m \geq 0$. This identity can be understood by noting that the summand is the generating function of partitions with Durfee rectangle of width $n + m$ and height $n - m$. Hence summing over $n \geq m$ yields the generating function of all partitions.

Returning to the q -supernomial of equation (2.9) and to lemma 2.5, we note that

$$q^{\sum_{k=2}^N j_{k-1}(L_k + \dots + L_N - j_k)} \begin{bmatrix} L_N \\ j_N \end{bmatrix} \begin{bmatrix} L_{N-1} + j_N \\ j_{N-1} \end{bmatrix} \cdots \begin{bmatrix} L_1 + j_2 \\ j_1 \end{bmatrix}$$

is the generating function of partitions with at most $j_1 + \dots + j_N$ parts, no part exceeding $L_1 + \dots + L_N - j_1$, with $N - 1$ successive Durfee rectangles of heights j_1, \dots, j_{N-1} and widths $L_2 + \dots + L_N - j_2, L_3 + \dots + L_N - j_3, \dots, L_N - j_N$. If we denote the set of such partitions by $S_{\mathbf{L}; \mathbf{j}}$ we get, using our graphical notation for restricted partition sets,



Sending $L_m \rightarrow \infty$, the generating function of $S_{\mathbf{L};j}$ is non-vanishing only when $j_1 = \dots = j_{m-1} = 0$. But in that case $\lim_{L_m \rightarrow \infty} S_{\mathbf{L};j}$ becomes the set of partitions with at most $j_m + \dots + j_N$ parts, $N - m$ successive Durfee rectangles of heights j_m, \dots, j_{N-1} and widths $L_{m+1} + \dots + L_N - j_{m+1}, \dots, L_N - j_N$. Hence the set

$$\lim_{L_m \rightarrow \infty} \bigcup_{\substack{j_m + \dots + j_N = a \\ j_1, \dots, j_{m-1} = 0}} S_{\mathbf{L};j}$$

corresponds to the set of all partitions with at most a parts, so that

$$\lim_{L_m \rightarrow \infty} \left[a - \frac{\ell_N}{2} \right] = \frac{1}{(q)_a}.$$

4 The Andrews–Gordon identities and q -supernomials

We now come to the second part of our paper where it is shown that several well-known q -series identities have a natural extension using the q -supernomials. The first and simplest example is Andrews' analytic counterpart of Gordon's partition theorem [4], for which we give polynomial analogues depending on multiple finitization parameters.

4.1 Brief history

To put things in a somewhat broader perspective, we first recall some facts about the Rogers–Ramanujan identities. For $a = 0, 1$ define the function

$$f(x; q) = f(x) = \sum_{n \in \mathbb{Z}} \frac{q^{n(n+a)} x^{2n}}{(x)_{n+1}}. \quad (4.1)$$

Obviously,

$$\lim_{x \rightarrow 1^-} (1-x)f(x; q) = \sum_{n \geq 0} \frac{q^{n(n+a)}}{(q)_n},$$

which equals one side of the Rogers–Ramanujan identities. Now view f as the generating function for polynomials, setting

$$f(x) = \sum_{M \in \mathbb{Z}} P(M) x^M.$$

By the q -binomial theorem [5]

$$\frac{1}{(x)_{n+1}} = \sum_{j \in \mathbb{Z}} \begin{bmatrix} j+n \\ j \end{bmatrix} x^j \quad (4.2)$$

it is readily seen that

$$P(M) = \sum_{n \in \mathbb{Z}} q^{n(n+a)} \begin{bmatrix} M-n \\ M-2n \end{bmatrix}. \quad (4.3)$$

For $M \geq 0$ these polynomials were introduced by MacMahon [36] and in this case the right-hand side of (4.3) corresponds to the generating function of partitions with difference between parts at least two, no parts exceeding $M + a - 1$ and all parts exceeding a .

An altogether different representation for P was found by Schur [43],

$$P(M) = \sum_{j=-\infty}^{\infty} (-1)^j q^{(5j+2a+1)j/2} \left[\begin{matrix} M+a \\ \lfloor \frac{1}{2}(M-5j) \rfloor \end{matrix} \right], \quad (4.4)$$

with $\lfloor \cdot \rfloor$ the integer part function. Equating (4.3) and (4.4) yields polynomial identities due to Andrews [2]. Letting $M \rightarrow \infty$ (for $|q| < 1$), the Rogers–Ramanujan identities are reproduced.

Apart from the fact that the function f in (4.1) acts as the generating function for the polynomials P , it has some further interesting properties [7]. Here we mention the q -difference equation

$$(1-x)f(x) = q^{a+1}x^2f(qx) \quad (4.5)$$

which implies the recurrences

$$P(M) = P(M-1) + q^{M+a-1}P(M-2).$$

It is indeed these recurrences and the initial conditions $P(0) = P(1) = 1$ which lead to the proof that (4.3) and (4.4) are equal.

4.2 The multivariable case

In ref. [7] it was suggested that the following multivariable function might be a relevant generalization of f :

$$f(\mathbf{x}) = \sum_{n_1, \dots, n_k \in \mathbb{Z}} \frac{q^{N_1^2 + \dots + N_k^2 + N_a + \dots + N_k} x_1^{2N_1} x_2^{2N_1 + 2N_2} \dots x_k^{2N_1 + \dots + 2N_k}}{(x_1)_{n_1+1} (x_2)_{n_2+1} \dots (x_k)_{n_k+1}}, \quad (4.6)$$

with

$$N_j = n_j + \dots + n_k,$$

$\mathbf{x} = (x_1, \dots, x_k)$ and $1 \leq a \leq k+1$. For $k=2$ this function has been analysed in relation with a generalization of the hard-hexagon model [9]. It is easy to show that f obeys the q -difference equations

$$(1-x_p)f(\mathbf{x}) = q^{2p-\min\{a-1,p\}} \left(\prod_{j=1}^k x_j^{2\min\{j,p\}} \right) f(x_1, \dots, qx_p, \dots, x_k), \quad (4.7)$$

for $p = 1, \dots, k$. These equations generalize (4.5).

Whereas (4.1) is closely related to the Rogers–Ramanujan identities, (4.6) has a connection to the Andrews–Gordon identities [4] given by

Theorem 4.1. *For $k \geq 1$ and $1 \leq a \leq k+1$,*

$$\sum_{n_1, \dots, n_k \geq 0} \frac{q^{N_1^2 + \dots + N_k^2 + N_a + \dots + N_k}}{(q)_{n_1} \dots (q)_{n_k}} = \frac{1}{(q)_{\infty}} \sum_{j=-\infty}^{\infty} (-1)^j q^{((2k+3)(j+1)-2a)j/2} = \prod_{\substack{j=1 \\ j \not\equiv 0, \pm a \pmod{2k+1}}}^{\infty} (1-q^j)^{-1}. \quad (4.8)$$

Indeed we see that $\lim_{\mathbf{x} \rightarrow (1^-, \dots, 1^-)} (1-x_1) \dots (1-x_k) f(\mathbf{x})$ returns the left-hand side of (4.8). Again viewing f as a generating function for polynomials, we define P as

$$f(\mathbf{x}) = \sum_{\mathbf{M} \in \mathbb{Z}^k} P(\mathbf{M}) x_1^{M_1} \dots x_k^{M_k}. \quad (4.9)$$

Multiple application of the q -binomial theorem (4.2) shows that for all $\mathbf{M} \in \mathbb{Z}^k$,

$$P(\mathbf{M}) = \sum_{n_1, \dots, n_k \in \mathbb{Z}} q^{N_1^2 + \dots + N_k^2 + N_a + \dots + N_k} \prod_{j=1}^k \begin{bmatrix} M_j + n_j - 2 \sum_{m=1}^j N_m \\ M_j - 2 \sum_{m=1}^j N_m \end{bmatrix}. \quad (4.10)$$

Less obvious is the following alternative form for P , generalizing (4.4) and starring the q -supernomials.

Theorem 4.2. *Set $\mathbf{L} = T\mathbf{M} - \mathbf{e}_{a-1} + \mathbf{e}_k \in \mathbb{Z}_+^k$ and $\boldsymbol{\ell} = T^{-1}\mathbf{L}$ with T defined in section 2.1 with N replaced by k . Then for $k \geq 1$ and $1 \leq a \leq k+1$,*

$$P(\mathbf{M}) = \sum_{j=-\infty}^{\infty} \left\{ q^{j((2k+3)(2j+1)-2a)} \begin{bmatrix} \mathbf{L} \\ \frac{b-a}{2} + (2k+3)j \end{bmatrix} - q^{(2j+1)((2k+3)j+a)} \begin{bmatrix} \mathbf{L} \\ \frac{b+a}{2} + (2k+3)j \end{bmatrix} \right\}, \quad (4.11)$$

with $b = k+1$ if $\ell_k + a + k$ is odd and $b = k+2$ if $\ell_k + a + k$ is even.

Before proving this theorem we note that equating the two different representations for P indeed yields a polynomial analogue of the Andrews–Gordon identity (4.8) based on k finitization parameters. Taking the limit $L_p \rightarrow \infty$ ($p = 1, \dots, k$) of $P(\mathbf{M})$ in the representation (4.11) reduces to the middle form of (4.8) by lemma 2.5. The same limit with $P(\mathbf{M})$ in representation (4.10) reduces to the left-hand side of (4.8), recalling that $\mathbf{M} = T^{-1}(\mathbf{L} + \mathbf{e}_{a-1} - \mathbf{e}_k)$.

We also remark that for $\mathbf{L} = L\mathbf{e}_1$, the polynomial identities implied by theorem 4.2 were proven by Foda and Quano [25] and by Kirillov [34]. (For $k = 1$ these are Andrews' identities discussed in section 4.1.) In ref. [47] (a slight variation of) the polynomial identities implied by the theorem were conjectured for all $\mathbf{L} = L\mathbf{e}_p$ ($p = 2, \dots, k$), and proven for $\mathbf{L} = L\mathbf{e}_k$.

Proof of theorem 4.2. We show that both (4.10) and (4.11) satisfy the recurrences

$$P(\mathbf{M}) = P(\mathbf{M} - \mathbf{e}_p) + q^{M_p - \min\{a-1, p\}} P(\mathbf{M} - 2T^{-1}\mathbf{e}_p), \quad (4.12)$$

for $p = 1, \dots, k-1$ and all $\mathbf{M} \in \mathbb{Z}^k$. The representation (4.10) satisfies this recurrence due to (4.7) and (4.9). To show that (4.11) obeys (4.12), we set $P(\mathbf{M}) = P(T^{-1}(\mathbf{L} + \mathbf{e}_{a-1} - \mathbf{e}_k)) = \tilde{P}(\mathbf{L})$ which implies

$$\tilde{P}(\mathbf{L}) = \tilde{P}(\mathbf{L} + \mathbf{e}_{p-1} - 2\mathbf{e}_p + \mathbf{e}_{p+1}) + q^{\ell_p - p} \tilde{P}(\mathbf{L} - 2\mathbf{e}_p), \quad (4.13)$$

for $\mathbf{L} \in \mathbb{Z}^k$, $\boldsymbol{\ell} = T^{-1}\mathbf{L}$ and $p = 1, \dots, k-1$. The proof that (4.11) satisfies (4.13) follows immediately from (2.18).

Thanks to these recurrences and lemma 2.4 it suffices to show that equations (4.10) and (4.11) are equal for the initial condition $\mathbf{L} = L\mathbf{e}_1$ with $L \geq 0$. This, however, is the one-variable polynomial identity proven in refs. [25, 34]. \square

Having established theorem 4.2, we wish to make a remark about the proof. In order to establish the equivalence between the representations (4.10) and (4.11) for $P(\mathbf{M})$, we have used the recurrences (4.12) for $p = 1, \dots, k - 1$ plus the one-parameter family of polynomial identities of refs. [25, 34]. Of course this second ingredient is a rather non-trivial result and one may wonder whether theorem 4.2 can be obtained without it. Indeed we point out that both (4.10) and (4.11) satisfy the recurrence (4.12) when $p = k$. Since this means that we have k independent recurrence relations it now suffices to take only a finite number of initial points to prove the theorem. However, admittedly, it is a non-trivial task to determine a suitable set of initial points, due to the “non-orthogonal” nature of the recurrences. To prove that (4.11) satisfies the $p = k$ recurrence, one needs to employ the formulae (3.10) and (3.11) (or rather, their corresponding $q \rightarrow 1/q$ analogue). Treating the $b = k + 1$ and $b = k + 2$ cases separately, the proof is a matter of straightforward algebra.

To conclude this section we mention that the function P in the representation of equation (4.10) has been extensively studied by Bressoud in refs. [19, 20] in his work on the O’Hara–Zeilberger identity. One of the results established in ref. [19] is a partition theoretic interpretation of $P(\mathbf{M})$ when $\mathbf{M} \in \mathbb{Z}_+^k$, obtained from theorem 2 therein by dropping the restriction that the partitions should have j parts and by replacing n_r by M_r ($r = 1, \dots, k$). (Note that in theorem 2 of ref. [19] it should be added that the number of parts is j and $ns(B)$ should be corrected to $n_s(B)$.) A second result from ref. [19] is an interpretation of $P(\mathbf{M})$ in terms of lattice paths, obtained from lemma 4 therein by replacing n_r by M_r ($r = 1, \dots, k$) and summing over $m_1, \dots, m_k \geq 0$.

5 Supernomial boson–fermion identities and continued fractions

5.1 (\mathbf{m}, \mathbf{n}) -Systems

In the previous section it was demonstrated how the q -supernomials can be applied to obtain polynomial identities related to identities of the Rogers–Ramanujan-type. In the remainder of this paper we treat a much more general application of the q -supernomials, which takes the results of section 4 as special case. In particular we show that to each pair of integers p and k , with $0 < 2k < p$ and $\gcd(p, k) = 1$, there corresponds a family of polynomial identities depending on N variables, with N restricted by $0 < N < (p - 1)/k - 1$.

Due to the complexity of this general result, we need a more systematic way to express our formulae, and to motivate the notation needed, we first recast some of our previous expressions in different forms, which have their genesis in the recent literature on “boson–fermion” identities (see ref. [17] and references therein).

Returning to the representation (4.10) of the function P , we replace k by N and define an N -dimensional vector $\mathbf{n} = (n_1, \dots, n_N)$ and a “companion” vector $\mathbf{m} = (m_1, \dots, m_N)$, such that $m_j = M_j - 2 \sum_{\ell=1}^j N_\ell$. Clearly, given \mathbf{M} , the vector \mathbf{m} determines the vector \mathbf{n} and vice versa. If we now eliminate the variables N_j in the quadratic exponent of q , equation (4.10) takes the form

$$P(\mathbf{M}) = \sum_{\mathbf{n} \in \mathbb{Z}^N} q^{\mathbf{n}T^{-1}(\mathbf{n} + \mathbf{e}_N - \mathbf{e}_{a-1})} \prod_{j=1}^N \begin{bmatrix} m_j + n_j \\ m_j \end{bmatrix},$$

where T is the Cartan-type matrix defined in section 2.1. Using $T\mathbf{M} = \mathbf{L} + \mathbf{e}_{a-1} - \mathbf{e}_N$, the

relation between \mathbf{n} and \mathbf{m} can be rewritten in the canonical form of an “ (\mathbf{m}, \mathbf{n}) -system” [13],

$$\mathbf{m} + \mathbf{n} = \frac{1}{2}(\mathcal{I}_T \mathbf{m} + \mathbf{e}_{a-1} - \mathbf{e}_N + \mathbf{L}).$$

Here \mathcal{I}_T is the incidence matrix of the tadpole graph with N nodes, $(\mathcal{I}_T)_{i,j} = \delta_{|i-j|,1} + \delta_{i,j}\delta_{i,N}$.

Now replacing $q \rightarrow 1/q$ and using equations (2.2) and (2.10), the result of theorem 4.2 translates to the following polynomial identity

$$\begin{aligned} & q^{\frac{1}{4}(N-a+1)} \sum_{\mathbf{n} \in \mathbb{Z}^N} q^{\frac{1}{4}\mathbf{m}T\mathbf{m} - \frac{1}{2}(m_{a-1} - m_N)} \prod_{j=1}^N \begin{bmatrix} m_j + n_j \\ m_j \end{bmatrix} \\ &= q^{\frac{1}{4N}(b-a)^2} \sum_{j=-\infty}^{\infty} \left\{ q^{\frac{j}{N}(3pj+p(b-N)-3a)} T(\mathbf{L}, \frac{b-a}{2} + pj) - q^{\frac{1}{N}(pj+a)(3j+b-N)} T(\mathbf{L}, \frac{b+a}{2} + pj) \right\}, \end{aligned} \tag{5.1}$$

for $\mathbf{L} \in \mathbb{Z}_+^N$, $p = 2N + 3$, $a = 1, \dots, N + 1$ ($m_0 = 0$) and $b = N + 1$ for $\ell_N + a + N$ odd and $b = N + 2$ for $\ell_N + a + N$ even, where $\ell = T^{-1}\mathbf{L}$.

In the following we generalize the above polynomial identity by considering more complicated incidence and Cartan-type matrices, defined through the continued fraction expansion of p/k , the simple case (5.1) corresponding to $p = 2N + 3$ and $k = 2$. Before doing so, we first discuss identity (5.1), and its subsequent generalizations, in the context of solvable lattice models and conformal field theories.

5.2 Conformal field theories, solvable lattice models and boson–fermion identities

For each $\mathbf{L} \in \mathbb{Z}_+^N$, equation (5.1) yields a polynomial identity. If we restrict \mathbf{L} to the lines $\mathbf{L} = \mathbf{L}\mathbf{e}_m$ with $1 \leq m \leq N$, the polynomials generated by (5.1) correspond to the one-dimensional configuration sums of solvable lattices models of Date et al. [21, 22] when their results are extended to “non-physical” regimes. (For $N = 1$ (5.1) corresponds to the 4-state Andrews–Baxter–Forrester model in regime IV [11].)

As was pointed out by Date et al., taking the limit $L \rightarrow \infty$ of the one-dimensional configuration sums yields branching functions or characters of (coset) conformal field theories. The one-dimensional configuration sums are thus polynomial approximations of conformal characters, and are often referred to as “finitized” characters. The q -supernomial identities involve higher-dimensional \mathbf{L} , providing a unification of various different “finitized” characters. In fact, thanks to the multi-dimensionality of \mathbf{L} , we can study much more general q -series limits (by sending an arbitrary number of components of \mathbf{L} to infinity), providing identities for generalizations of the $A_1^{(1)}$ branching functions, see section 5.6.

In the physics literature identities such as (5.1) are known as (polynomial) boson–fermion identities, the left-hand side involving the (\mathbf{m}, \mathbf{n}) -system being “fermionic” and the alternating-sign expression on the right-hand side being “bosonic”. Indeed, the left-hand side admits an interpretation as the partition function for a system of quasiparticles with fractional statistics, obeying Fermi’s exclusion principle [32, 33]. The (\mathbf{m}, \mathbf{n}) -system of a fermionic function then describes the combinatorics of the fermionic quasiparticles, the j -th components of \mathbf{n} and \mathbf{m} having the interpretation of occupation number of particles and antiparticles of charge j , respectively.

The term bosonic has its origin in conformal field theory (CFT). Each CFT is described by an underlying symmetry algebra, such as the Virasoro algebra, the superconformal algebra or other extended algebras. The states in a CFT can be determined from the highest weight representation of the symmetry algebra by constructing the Verma modules, each of which consists of a highest weight state and its descendants, obtained by acting with the lowering operators on this state. Some descendants may have zero norm and are referred to as singular vectors. For the calculation of the characters or branching functions the contribution of these singular vectors has to be subtracted out. This procedure is called the Feigin and Fuchs construction [23, 24] (and is analogous to the sieving technique of refs. [3, 5]). Since it relies on the use of a “bosonic Fock space”, the resulting character expressions are referred to as bosonic. Even though the Feigin–Fuchs construction applies to conformal characters and not their polynomial finitizations, the term bosonic is now generally used for expressions of alternating-sign-type, such as the right side of (5.1).

When taking the finitization parameter(s) in polynomial boson–fermion identities to infinity, one obtains q -series identities of boson–fermion-type. In certain special cases, the bosonic side can be further rewritten using the triple product identity or its generalizations, to yield identities of Rogers–Ramanujan type. The term boson–fermion-type is thus referring to a wider class of identities than Rogers–Ramanujan-type.

We finally wish to emphasize that polynomial boson–fermion identities which depend on a vector \mathbf{L} rather than a scalar L have not appeared in the literature before, and it is rather intriguing to note that many of the existing polynomial identities related to configuration sums of different solvable lattive models, can be cast in one unifying form, using the q -supernomials.

5.3 Continued fraction expansion and Takahashi–Suzuki decomposition

After the previous intermezzo we describe how (5.1) can be generalized by considering the continued fraction expansion of p/k . All notation and definitions of this subsection are borrowed from the work of Berkovich and McCoy [14].

Definition 5.1. *Let p, k be integers such that $0 < 2k < p$ and $\gcd(p, k) = 1$. Then the integers n and ν_j ($0 \leq j \leq n$) are defined by the continued fraction expansion*

$$\frac{p}{k} = 1 + \nu_0 + \frac{1}{\nu_1 + \frac{1}{\nu_2 + \dots + \frac{1}{\nu_n + 2}}}. \quad (5.2)$$

When $k = 1$, we have $n = 0$ and the continued fraction expansion (5.2) reads $p = \nu_0 + 3$. Using n and ν_j ($0 \leq j \leq n$) we introduce the sums

$$t_m = \sum_{j=0}^{m-1} \nu_j \quad (1 \leq m \leq n+1) \quad \text{and} \quad t_0 = -1, \quad (5.3)$$

which define a fractional incidence and Cartan-type matrix as follows.

Definition 5.2. Let n, ν_j ($0 \leq j \leq n$) and t_m ($0 \leq m \leq n+1$) be given by the continued fraction expansion of p/k . The fractional incidence matrix \mathcal{I}_B is given by

$$(\mathcal{I}_B)_{i,j} = \begin{cases} \delta_{i,j+1} + \delta_{i,j-1} & \text{for } 1 \leq i < t_{n+1}, i \neq t_m, \\ \delta_{i,j+1} + \delta_{i,j} - \delta_{i,j-1} & \text{for } i = t_m, 1 \leq m \leq n - \delta_{\nu_n,0}, \\ \delta_{i,j+1} + \delta_{\nu_n,0}\delta_{i,j} & \text{for } i = t_{n+1}. \end{cases} \quad (5.4)$$

\mathcal{I}_B defines a Cartan-type matrix B via $B = 2I - \mathcal{I}_B$, with I the t_{n+1} by t_{n+1} unit matrix.

Notice that when $k = 1$, the incidence matrix \mathcal{I}_B has components $(\mathcal{I}_B)_{i,j} = \delta_{|i-j|,1}$ ($i, j = 1, \dots, p-3$), so that B corresponds to the Cartan matrix of the Lie algebra A_{p-3} .

For $0 \leq m \leq n$, we set the recurrences

$$y_{m+1} = y_{m-1} + (\nu_m + 2\delta_{m,n} + \delta_{m,0})y_m, \quad y_{-1} = 0, \quad y_0 = 1, \quad (5.5)$$

$$\bar{y}_{m+1} = \bar{y}_{m-1} + (\nu_m + 2\delta_{m,n} + \delta_{m,0})\bar{y}_m, \quad \bar{y}_{-1} = -1, \quad \bar{y}_0 = 1, \quad (5.6)$$

so that $y_{n+1} = p$ and $\bar{y}_{n+1} = p - k$. This leads to the following definition.

Definition 5.3. Let n, t_m, y_m and \bar{y}_m be defined by the continued fraction expansion of p/k . Then the Takahashi lengths l_{j+1} and truncated Takahashi lengths \bar{l}_{j+1} are defined as

$$\left. \begin{aligned} l_{j+1} &= y_{m-1} + (j - t_m)y_m \\ \bar{l}_{j+1} &= \bar{y}_{m-1} + (j - t_m)\bar{y}_m \end{aligned} \right\} \text{ for } 0 \leq m \leq n, t_m < j \leq t_{m+1} + \delta_{n,m}. \quad (5.7)$$

Finally, vectors $\mathbf{Q}^{(j)}$ ($j = 1, \dots, t_{n+1} + 1$) are needed to specify parities of summation variables. For $1 \leq i \leq t_{n+1}$ and $0 \leq m \leq n$ such that $t_m < j \leq t_{m+1} + \delta_{m,n}$ the components of $\mathbf{Q}^{(j)}$ are recursively defined as

$$Q_i^{(j)} = \begin{cases} 0 & \text{for } j \leq i \leq t_{n+1}, \\ j - i & \text{for } t_m \leq i < j, \\ Q_{i+1}^{(j)} + Q_{t_{m'}+1}^{(j)} & \text{for } t_{m'-1} \leq i < t_{m'}, 1 \leq m' \leq m, \end{cases} \quad (5.8)$$

with $Q_{t_{n+1}}^{(t_{n+1}+1)} = 0$ for $\nu_n = 0$.

In the next section we repeatedly use the above definitions and it will be convenient to refer to all of the equations of this section as “the Takahashi–Suzuki (TS) decomposition of p/k ” [44].

5.4 The supernomial boson–fermion identities

We are now prepared for the polynomial boson–fermion identities based on the continued fraction expansion of p/k , and depending on multiple finitization parameters. To this end let us first define the generalizations of the bosonic and fermionic side of (5.1), respectively. The bosonic side involves the supernomials $T(\mathbf{L}, a)$ of equation (2.10).

Definition 5.4. Consider the TS decomposition of p/k with p and k positive integers and $\gcd(p, k) = 1$. Fix a positive integer N such that $N < (p-1)/k - 1$ and let $\mathbf{L} \in \mathbb{Z}^N$, $\boldsymbol{\ell} = T^{-1}\mathbf{L}$.

Choose integers a and b such that $a + b + \ell_N$ is even and such that $a = l_{\alpha+1}$ and $b = l_{\beta+1} \geq 2$ are Takahashi lengths. Then

$$B_{a,b}^{(p,k,N)}(\mathbf{L}) = q^{\frac{1}{4N}(b-a)^2} \sum_{j=-\infty}^{\infty} \left\{ q^{\frac{j}{N}(p(p-kN)j+pr-(p-kN)a)} T\left(\mathbf{L}, \frac{b-a}{2} + pj\right) - q^{\frac{1}{N}(pj+a)((p-kN)j+r)} T\left(\mathbf{L}, \frac{b+a}{2} + pj\right) \right\}, \quad (5.9)$$

with $r = b - N(b - \bar{b})$ and $\bar{b} = \bar{l}_{\beta+1}$.

Definition 5.5. Fix all parameters as in definition 5.4. Then

$$F_{a,b}^{(p,k,N)}(\mathbf{L}) = q^{\Delta_{a,b}} \sum_{\substack{\mathbf{m} \in \mathbb{Z}_+^{t_{n+1}} \\ \mathbf{m} \equiv \mathbf{Q}_{a,b} \pmod{2}}} q^{\frac{1}{4}\mathbf{m}B\mathbf{m} - \frac{1}{2}\mathbf{A}_{a,b}\mathbf{m}} \prod_{j=1}^{t_{n+1}} \begin{bmatrix} m_j + n_j \\ m_j \end{bmatrix}. \quad (5.10)$$

Here the (\mathbf{m}, \mathbf{n}) -system is given by

$$\mathbf{m} + \mathbf{n} = \frac{1}{2} \left(\mathcal{I}_B \mathbf{m} + \mathbf{u}_a + \mathbf{u}_b + \sum_{i=1}^N L_i \mathbf{e}_i \right), \quad (5.11)$$

where

$$\begin{aligned} \mathbf{u}_a &= \mathbf{e}_\alpha - \sum_{i=m+1}^n \mathbf{e}_{t_i} \quad \text{for } t_m < \alpha \leq t_{m+1} + \delta_{m,n}, \\ \mathbf{u}_b &= \mathbf{e}_\beta - \sum_{i=m'+1}^n \mathbf{e}_{t_i} \quad \text{for } t_{m'} < \beta \leq t_{m'+1} + \delta_{m',n}. \end{aligned} \quad (5.12)$$

In addition, for $t_i < j \leq t_{i+1}$ ($i = 0, \dots, n$),

$$(\mathbf{A}_{a,b})_j = \begin{cases} (\mathbf{u}_b)_j & \text{for } i \text{ odd,} \\ (\mathbf{u}_a)_j & \text{for } i \text{ even.} \end{cases} \quad (5.13)$$

The restriction $\mathbf{m} \equiv \mathbf{Q}_{a,b} \pmod{2}$ on the sum is an abbreviation of $m_i \equiv (\mathbf{Q}_{a,b})_i \pmod{2}$ for all $i = 1, \dots, t_{n+1}$. Here

$$\mathbf{Q}_{a,b} = \sum_{j=1}^{t_{n+1}} (\mathbf{u}_a + \mathbf{u}_b)_j \mathbf{Q}^{(j)} + (\delta_{\alpha, t_{n+1}+1} + \delta_{\beta, t_{n+1}+1}) \mathbf{Q}^{(t_{n+1}+1)} + \sum_{j=1}^N L_j \mathbf{Q}^{(j)}, \quad (5.14)$$

with vectors $\mathbf{Q}^{(j)}$ defined in equation (5.8). Finally, $\Delta_{a,b}$ is fixed through the condition

$$q^{-\frac{1}{4N}(b-a)^2} F_{a,b}^{(p,k,N)}(\mathbf{L}) \Big|_{q=0} = 1 \quad \text{for } \ell_N \geq |b-a| \text{ and } \mathbf{L} \in \mathbb{Z}_+^N. \quad (5.15)$$

Before proceeding, let us remark that the parity restriction $\mathbf{m} \equiv \mathbf{Q}_{a,b} \pmod{2}$ can be derived from $m_{t_{n+1}} \equiv \delta_{\alpha, t_{n+1}+1} + \delta_{\beta, t_{n+1}+1} \pmod{2}$ and the condition that \mathbf{n} in (5.11) is a vector with integer components.

For $\mathbf{L} = L\mathbf{e}_1$ ($L \geq 0$), the bosonic function in (5.9) corresponds (up to a prefactor) to the one-dimensional configuration sum of the Andrews–Baxter–Forrester (ABF) model [11] as obtained in ref. [27]. This follows from equation (2.12), yielding

$$B_{a,b}^{(p,k,N)}(L\mathbf{e}_1) = q^{\frac{1}{4}(b-a)^2} \sum_{j=-\infty}^{\infty} \left\{ q^{j(p-k)j+p\bar{b}-(p-k)a} \left[\frac{L}{\frac{L+b-a}{2}} + pj \right] - q^{(pj+a)((p-k)j+\bar{b})} \left[\frac{L}{\frac{L+b+a}{2}} + pj \right] \right\}. \quad (5.16)$$

The fermionic function in (5.10) for $\mathbf{L} = L\mathbf{e}_1$ corresponds to the Berkovich–McCoy finitization [14] of the Virasoro characters of the minimal model $M(p-k, p)$.

Actually, in ref. [27] two functions $D_L(a, b, b+1)$ and $D_L(a, b, b-1)$ have been considered, with (5.16) corresponding to $D_L(a, b, b-1)$. They are however not independent, but related by the symmetry $D_L(a, b, b+1) = D_L(p-a, p-b, p-b-1)$ and hence it is sufficient to restrict attention to generalizations of $D_L(a, b, b-1)$. We should also mention that in ref. [27] $D_L(a, b, b-1)$ is defined for all $1 \leq a \leq p-1$ and $2 \leq b \leq p-1$ and not just for a, b being Takahashi lengths. The reason for restricting to Takahashi lengths is that in the more general case the fermionic functions become rather complicated [17].

The following theorem claims a polynomial boson–fermion type relation.

Theorem 5.1. *For $b = l_{\beta+1}$ with $\beta \geq N$ and $\mathbf{L} \in \mathbb{Z}_+^N$, the functions $B_{a,b}^{(p,k,N)}(\mathbf{L})$ and $F_{a,b}^{(p,k,N)}(\mathbf{L})$ of definitions 5.4 and 5.5 satisfy the identity*

$$B_{a,b}^{(p,k,N)}(\mathbf{L}) = F_{a,b}^{(p,k,N)}(\mathbf{L}). \quad (5.17)$$

Proof. We show that both sides of (5.17) satisfy the recursion relation

$$X(\mathbf{L}) = X(\mathbf{L} - 2\mathbf{e}_i) + q^{\frac{1}{2}(L_i-1)} X(\mathbf{L} + \mathbf{e}_{i-1} - 2\mathbf{e}_i + \mathbf{e}_{i+1}), \quad (5.18)$$

for $1 \leq i \leq N-1$. Since in ref. [17] identity (5.17) has been shown to hold for $\mathbf{L} = L\mathbf{e}_1$ with $L \in \mathbb{Z}_+$ and since $q^{\frac{1}{4}LT^{-1}L - \frac{a^2}{N}} X(\mathbf{L}; 1/q)$ satisfies (2.20), this establishes theorem 5.1 by lemma 2.4.

The recurrences (2.19) for the supernomials immediately imply that $B_{a,b}^{(p,k,N)}(\mathbf{L})$ satisfies (5.18). In order to show that $F_{a,b}^{(p,k,N)}(\mathbf{L})$ satisfies (5.18), we apply the q -binomial recurrence (2.4) to the term $j = i$ of the product in (5.10),

$$\prod_{j=1}^{t_{n+1}} \begin{bmatrix} m_j + n_j \\ m_j \end{bmatrix} = \prod_{j=1}^{t_{n+1}} \begin{bmatrix} m_j + n_j - \delta_{i,j} \\ m_j \end{bmatrix} + q^{n_i} \prod_{j=1}^{t_{n+1}} \begin{bmatrix} m_j + n_j - \delta_{i,j} \\ m_j - \delta_{i,j} \end{bmatrix}.$$

The first term on the right-hand side directly yields $F_{a,b}^{(p,k,N)}(\mathbf{L} - 2\mathbf{e}_i)$. In the second term we change the summation variable $m_i \rightarrow m_i + 1$. Since \mathbf{n} can be expressed in terms of \mathbf{m} as

$$n_j = \frac{1}{2} (-(B\mathbf{m})_j + (\mathbf{u}_a)_j + (\mathbf{u}_b)_j + \theta(j \leq N)L_j),$$

this variable change gives

$$\frac{1}{4} \mathbf{m} B \mathbf{m} - \frac{1}{2} \mathbf{A}_{a,b} \mathbf{m} + n_i \rightarrow \frac{1}{2} (L_i - \frac{1}{2} B_{i,i}) + \frac{1}{2} (\mathbf{u}_a + \mathbf{u}_b - \mathbf{A}_{a,b})_i + \frac{1}{4} \mathbf{m} B \mathbf{m} - \frac{1}{2} \mathbf{A}_{a,b} \mathbf{m}.$$

Now observe that the condition $N < (p-1)/k - 1$ implies that $N \leq \nu_0$, so that $i < \nu_0$, with ν_0 defined in (5.2). This means that $B_{i,i} = 2$ and $(\mathbf{u}_a + \mathbf{u}_b - \mathbf{A}_{a,b})_i = (\mathbf{u}_b)_i = 0$ since $\beta \geq N$. Hence we see that after the variable change the second term yields $q^{\frac{1}{2}(L_i-1)} F_{a,b}^{(p,k,N)}(\mathbf{L} + \mathbf{e}_{i-1} - 2\mathbf{e}_i + \mathbf{e}_{i+1})$ and (5.18) is proven for $X = F$. \square

From the proof of theorem 5.1 follows that $\Delta_{a,b}$ in (5.10) is independent of N , and can therefore be determined by considering the case $N = 1$.¹ The explicit expression for $\Delta_{a,b}$ is quite involved and can be found in ref. [17].

Finally notice that for $X = B, F$ and $N \leq M$

$$X_{a,b}^{(p,k,M)}((L_1, \dots, L_N, 0, \dots, 0)) = X_{a,b}^{(p,k,N)}((L_1, \dots, L_N)). \quad (5.19)$$

For $X = B$ this identity is quite remarkable since, due to the fact that the superscript on the left-hand side involves M and the one on the right-hand side involves N , the quadratic exponents of q in the definition of the bosonic side (5.9) are different. Relation (5.19) for $X = B$ may, however, be deduced from property (2.11) of the q -supernomials. For $X = F$ relation (5.19) follows directly from the (\mathbf{m}, \mathbf{n}) -system (5.11) and the fact that $\Delta_{a,b}$ is N independent.

5.5 The case $k = 1$

The identities of section 5.4 are very general, but have the drawback of being rather implicit. This warrants elaborating the simple but important case $k = 1$.

As noted before, for $k = 1$ the TS decomposition is trivial, yielding $n = 0$, $p = \nu_0 + 3$ and $t_1 = p - 3$, so that $\mathcal{I}_B = \mathcal{I}$ and $B = C$ with \mathcal{I} and C the incidence matrix and Cartan matrix of A_{p-3} , respectively. The (truncated) Takahashi lengths are given by $l_{j+1} = j + 1$ and $\bar{l}_{j+1} = j$ for $0 \leq j \leq p - 2$. This implies the following simplifications for some of the quantities of definition 5.5; $\mathbf{A}_{a,b} = \mathbf{u}_a = \mathbf{e}_{a-1}$, $\mathbf{u}_b = \mathbf{e}_{b-1}$ and $\mathbf{Q}_{a,b} = \mathbf{Q}^{(a-1)} + \mathbf{Q}^{(b-1)} + \sum_{i=2}^N L_i \mathbf{Q}^{(i)}$ where $\mathbf{Q}^{(j)} = \mathbf{e}_{j-1} + \mathbf{e}_{j-3} + \dots$. Finally, we find that $\Delta_{a,b} = (b-a)/4$. We can thus conclude the following supernomial identity ($m_0 = 0$):

$$\begin{aligned} & \sum_{j=-\infty}^{\infty} \left\{ q^{\frac{j}{N}(p(p-N)j+pr-(p-N)a)} T(\mathbf{L}, \frac{b-a}{2} + pj) - q^{\frac{1}{N}(pj+a)((p-N)j+r)} T(\mathbf{L}, \frac{b+a}{2} + pj) \right\} \\ &= q^{\frac{1}{4N}(b-a)(a-b+N)} \sum_{\substack{\mathbf{m} \in \mathbb{Z}_+^{p-3} \\ \mathbf{m} \equiv \mathbf{Q}_{a,b} \pmod{2}}} q^{\frac{1}{4} \mathbf{m} C \mathbf{m} - \frac{1}{2} m_{a-1}} \prod_{j=1}^{p-3} \begin{bmatrix} m_j + n_j \\ m_j \end{bmatrix}, \quad (5.20) \end{aligned}$$

for $\mathbf{L} \in \mathbb{Z}_+^N$, $1 \leq a \leq p-1$, $N+1 \leq b \leq p-1$ and $r = b - N$. The corresponding (\mathbf{m}, \mathbf{n}) -system is given by

$$\mathbf{m} + \mathbf{n} = \frac{1}{2} \left(\mathcal{I} \mathbf{m} + \mathbf{e}_{a-1} + \mathbf{e}_{b-1} + \sum_{i=1}^N L_i \mathbf{e}_i \right).$$

For $\mathbf{L} = L \mathbf{e}_1$ ($L \geq 0$), equation (5.20), which has been proven in refs. [13, 45, 46], corresponds to an identity for the one-dimensional configuration sums of the Andrews–Baxter–Forrester

¹Equation (5.1) may seem to contradict this. Note however that in theorem 5.1 we can choose p, k and N independently. Equation (5.1) simply corresponds to the special choice $p = 2N + 3$, introducing N dependence in “ p -dependent” quantities.

model [11]. More generally, for $\mathbf{L} = L\mathbf{e}_N$, (5.20) is an identity for configuration sums of RSOS models of Date et al. [21, 22], and has first been proven in refs. [39, 40].

5.6 q -Series limits of theorem 5.1

5.6.1 Limits related to generalizations of the $\mathbf{A}_1^{(1)}$ branching functions

In this section we consider the limit $L_N \rightarrow \infty$ of the polynomials in (5.9) and (5.10), and show how this limit relates to the branching functions of the $\mathbf{A}_1^{(1)}$ cosets

$$\frac{(\mathbf{A}_1^{(1)})_N \times (\mathbf{A}_1^{(1)})_{N'}}{(\mathbf{A}_1^{(1)})_{N+N'}} \quad N \in \mathbb{Z}, \quad N' \in \mathbb{R}. \quad (5.21)$$

We denote the normalized branching functions of these cosets by $\hat{\chi}_{r,s;\ell}^{(P,P';N)}$ where

$$N' = \frac{NP}{P' - P} - 2 \quad \text{or} \quad N' = -2 - \frac{NP'}{P' - P},$$

with the restrictions $P < P'$, $P' - P \equiv 0 \pmod{N}$ and $\gcd(\frac{P'-P}{N}, P') = 1$.

Before presenting the q -series limits of the supernomial identities, we first define generalizations of the $\mathbf{A}_1^{(1)}$ branching functions.

Definition 5.6. *Let N, P, P' be positive integers such that $P < P'$, $P' - P \equiv 0 \pmod{N}$ and $\gcd(\frac{P'-P}{N}, P') = 1$. Fix $\sigma = 0, 1$, $\mathbf{L} \in \mathbb{Z}_+^{N-1}$ and choose $1 \leq r < P$ and $1 \leq s < P'$ such that $r - s + N(C^{-1}\mathbf{L})_{N-1} + N\sigma$ is even. Then*

$$\begin{aligned} \hat{\chi}_{r,s;\mathbf{L},\sigma}^{(P,P';N)}(q) &= q^{-\frac{\mathbf{L}C^{-1}\mathbf{L}}{2(N+2)}} \\ &\times \sum_{0 \leq m \leq N/2} c_{2m}^{\mathbf{L},\sigma}(q) \left(\sum_{\substack{j \in \mathbb{Z} \\ m_{r-s}(j) \equiv \pm m \pmod{N}}} q^{\frac{j}{N}(jPP' + P'r - Ps)} - \sum_{\substack{j \in \mathbb{Z} \\ m_{r+s}(j) \equiv \pm m \pmod{N}}} q^{\frac{j}{N}(jP'+s)(jP+r)} \right), \end{aligned} \quad (5.22)$$

where $m_a(j) = (a/2 + P'j)$. The sum over m runs over integers if $r - s$ is even and half-integers if $r - s$ is odd. The string-like function $c_{2m}^{\mathbf{L},\sigma}$ is defined in (2.29).

Setting $\mathbf{L} = \mathbf{e}_\ell$ and using (2.31), this reduces to the normalized branching functions of the cosets (5.21), in the representation of ref. [1] (for the unitary case $P' = P + N$, so that $N' \in \mathbb{Z}$, see also refs. [12, 31, 38]),

$$\hat{\chi}_{r,s;\ell}^{(P,P';N)}(q) = \begin{cases} \hat{\chi}_{r,s;\mathbf{e}_\ell,0}^{(P,P';N)}(q) & \text{for } 0 \leq \ell < N, \\ \hat{\chi}_{r,s;0,1}^{(P,P';N)}(q) & \text{for } \ell = N. \end{cases} \quad (5.23)$$

The generalized branching functions in definition 5.6 arise as the following limit of the bosonic polynomials of equation (5.9).

Lemma 5.1. *Fix $\sigma = 0, 1$ and positive integers N, M, p, k such that $1 \leq N \leq M < (p-1)/k - 1$ and $\gcd(p, k) = 1$, and define the TS decomposition of p/k . Let a, b, r and $B_{a,b}^{(p,k,M)}$ be as in definition 5.4, and $\mathbf{L} \in \mathbb{Z}_+^{N-1}$ such that $r - a + N(C^{-1}\mathbf{L})_{N-1} + N\sigma$ is even. Then for $|q| < 1$,*

$$\hat{\chi}_{r,a;\mathbf{L},\sigma}^{(p-kN,p;N)}(q) = q^{-\frac{1}{4N}(b-a)^2 - \frac{1}{4}\mathbf{L}C^{-1}\mathbf{L}} \lim_{\substack{L_N \rightarrow \infty \\ L_N \equiv \sigma \pmod{2}}} B_{a,b}^{(p,k,M)}((\mathbf{L}, L_N, 0, \dots, 0)). \quad (5.24)$$

Proof. Because of (5.19) with $X = B$, it is sufficient to establish (5.24) for $N = M$. From corollary 2.1 and the symmetries (2.30), equation (5.24) follows immediately. \square

Thanks to theorem 5.1 and lemma 5.1 we obtain the following fermionic representation for the extended branching functions.

Corollary 5.1. *Define all quantities as in lemma 5.1. Then*

$$\hat{\chi}_{r,a;\mathbf{L},\sigma}^{(p-kN,p;N)}(q) = q^{-\frac{1}{4N}(b-a)^2 - \frac{1}{4}LC^{-1}\mathbf{L} + \Delta_{a,b}} \sum_{\substack{\mathbf{m} \in \mathbb{Z}_+^{t_{n+1}} \\ \mathbf{m} \equiv \mathbf{Q}_{a,b} \pmod{2}}} \frac{q^{\frac{1}{4}\mathbf{m}B\mathbf{m} - \frac{1}{2}\mathbf{A}_{a,b}\mathbf{m}}}{(q)_{m_N}} \prod_{\substack{j=1 \\ j \neq N}}^{t_{n+1}} \begin{bmatrix} m_j + n_j \\ m_j \end{bmatrix},$$

with (\mathbf{m}, \mathbf{n}) -system $\mathbf{m} + \mathbf{n} = \frac{1}{2}(\mathcal{I}_B\mathbf{m} + \mathbf{u}_a + \mathbf{u}_b + \sum_{k=1}^{N-1} L_k \mathbf{e}_k)$, \mathbf{u}_a and \mathbf{u}_b as in (5.12) and $\mathbf{Q}_{a,b}$ as in (5.14) with L_N therein replaced by σ .

For all but the cases where the above corresponds to the $A_1^{(1)}$ branching functions, we believe this result to be new. For the special case of equation (5.23), the above was found in part in ref. [26] and in general in refs. [14, 17] for $N = 1$, in ref. [39] for $N \geq 2$ and $k = 1$, in ref. [47] for $N \geq 2$, $k = 2$ and $p = 2N + 3$ and in ref. [18] for general $N \geq 2$.

5.6.2 Further limits

We briefly discuss the more general limits $L_{k_i} \rightarrow \infty$ ($i = 1, \dots, h$) of theorem 5.1, where $1 \leq k_1 < k_2 < \dots < k_h \leq N$. As in definition 2.4, $K = \{k_1, \dots, k_h\}$ and $\bar{K} = \{1, \dots, N\} - K$. In addition we denote $K' = \{1, \dots, t_{n+1}\} - K$. Then we obtain

$$\lim_{\substack{L_{k_1}, \dots, L_{k_h} \rightarrow \infty \\ L_{k_i} \equiv \sigma_{k_i} \pmod{2}, (1 \leq i \leq h)}} F_{a,b}^{(p,k,N)}(\mathbf{L}) = q^{\Delta_{a,b}} \sum_{\substack{\mathbf{m} \in \mathbb{Z}_+^{t_{n+1}} \\ \mathbf{m} \equiv \mathbf{Q}_{a,b} \pmod{2}}} \frac{q^{\frac{1}{4}\mathbf{m}B\mathbf{m} - \frac{1}{2}\mathbf{A}_{a,b}\mathbf{m}}}{(q)_{m_{k_1}} \cdots (q)_{m_{k_h}}} \prod_{j \in K'} \begin{bmatrix} m_j + n_j \\ m_j \end{bmatrix}$$

where $\mathbf{m} + \mathbf{n} = \frac{1}{2}(\mathcal{I}_B\mathbf{m} + \mathbf{u}_a + \mathbf{u}_b + \sum_{k \in \bar{K}} L_k \mathbf{e}_k)$, \mathbf{u}_a and \mathbf{u}_b as in (5.12) and $\mathbf{Q}_{a,b}$ given by

$$\mathbf{Q}_{a,b} = \sum_{j=1}^{t_{n+1}} (\mathbf{u}_a + \mathbf{u}_b)_j \mathbf{Q}^{(j)} + (\delta_{\alpha, t_{n+1}+1} + \delta_{\beta, t_{n+1}+1}) \mathbf{Q}^{(t_{n+1}+1)} + \sum_{j \in K} \sigma_j \mathbf{Q}^{(j)} + \sum_{j \in \bar{K}} L_j \mathbf{Q}^{(j)},$$

recalling that $a = l_{\alpha+1}$ and $b = l_{\beta+1}$.

According to equation (2.32), the same limit, but now for $B_{a,b}^{(p,k,N)}$, is obtained by replacing the q -supernomials in (5.9) by $b_{2m}^{\{L_k | k \in \bar{K}\} \{\sigma_k | k \in K\}}$ with $m = (b-a)/2 + pj$ for the first q -supernomial and $m = (b+a)/2 + pj$ for the second q -supernomial.

A yet different set of q -series identities can be obtained from theorem 5.1 by first transforming $q \rightarrow 1/q$ before taking some (or all) of the components of \mathbf{L} to infinity. Recalling (2.10) we find that

$$q^{\frac{1}{4}L\mathbf{T}^{-1}\mathbf{L}} B_{a,b}^{(p,k,N)}(\mathbf{L}; 1/q) = \sum_{j=-\infty}^{\infty} \left\{ q^j (pkj + p(b-\bar{b}) - ka) \begin{bmatrix} \mathbf{L} \\ \frac{b-a}{2} + pj \end{bmatrix} - q^{(pj+a)(kj+b-\bar{b})} \begin{bmatrix} \mathbf{L} \\ \frac{b+a}{2} + pj \end{bmatrix} \right\},$$

where we have used that $r = b - N(b - \bar{b})$. Hence, by lemma 2.5, we see that for all $m = 1, \dots, N$,

$$\lim_{L_m \rightarrow \infty} q^{\frac{1}{4}LT^{-1}\mathbf{L}} B_{a,b}^{(p,k,N)}(\mathbf{L}; 1/q) = \frac{1}{(q)_\infty} \sum_{j=-\infty}^{\infty} \left\{ q^{j(pkj+p(b-\bar{b})-ka)} - q^{(pj+a)(kj+b-\bar{b})} \right\},$$

independent of N . The right-hand side of this expression is recognized as the bosonic form of the (normalized) Virasoro characters $\hat{\chi}_{b-\bar{b},a}^{(k,p)}(q)$.

Of course we can carry out a similar calculation for the function F of theorem 5.1, but since the resulting fermionic representations for the characters $\hat{\chi}_{b-\bar{b},a}^{(k,p)}(q)$ coincide with those given in refs. [14, 17], we refer the interested reader to the literature.

6 Discussion

We conclude with some final remarks about the results of this paper.

First we comment on the restriction $N < (p-1)/k - 1$ imposed on the q -supernomial identity of theorem 5.1. Recall the continued fraction expansion of p/k as given in equation (5.2). This expansion defines integers ν_0, \dots, ν_n which are used to define an (\mathbf{m}, \mathbf{n}) -system of dimension $\nu_0 + \dots + \nu_n$. As can be seen from equation (5.4), the components of the (\mathbf{m}, \mathbf{n}) -system with labels between $\nu_0 + \dots + \nu_i$ and $\nu_0 + \dots + \nu_{i+1}$ can be viewed as one unit. Each of these different units is referred to as a “zone” [44]. Now the restriction in our theorem implies that $N \leq \nu_0$ and hence that our recurrences essentially act within the zeroth zone of the (\mathbf{m}, \mathbf{n}) -system only. To obtain q -supernomial identities which are true for $N > \nu_0$ one needs to generalize the q -supernomials to satisfy the recurrences implied by the full set of zones of the (\mathbf{m}, \mathbf{n}) -system. Such modified q -supernomials would thus reflect the complete continued fraction expansion of p/k and not just its integer part. Indeed polynomial boson-fermion identities (related to the Gordon-Göllnitz identities) with finitization parameter L beyond zone zero have been found in refs. [16, 15]. It would be an interesting problem to generalize these to identities depending on multiple finitization parameters, using more general q -supernomials.

Another point of interest is the combinatorial interpretation of the bosonic and fermionic polynomials of equations (5.9) and (5.10). For $\mathbf{L} = L\mathbf{e}_i$ ($i = 1, \dots, N$) these polynomials correspond to one-dimensional configuration sums of solvable lattice models [11, 27, 21, 22]. In those cases we have an interpretation in terms of weighted lattice paths. Another interpretation, in terms of partitions with prescribed hook differences, exists for $\mathbf{L} = L\mathbf{e}_1$ [10]. For the general N -dimensional case, however, we have not been able to generalize either of these. What is possible is to give a simple interpretation in terms of lattice paths when $q = 1$. In fact, instead of working with lattice paths, we give an equivalent matrix formulation in the following.

Fix an integer $p \geq 4$ and define a family of incidence matrices A_0, \dots, A_{p-2} as follows:

$$A_k A_1 = A_{k-1} + A_{k+1} \quad \text{for } k = 1, \dots, p-3,$$

with $A_0 = I$ the $(p-1) \times (p-1)$ identity matrix and $A_1 = \mathcal{I}$ the incidence matrix of the Lie algebra A_{p-1} , with entries $\mathcal{I}_{a,b} = \delta_{|a-b|,1}$. The above set of matrices forms a commuting family. Some further notable properties are $A_{p-2} = Y$ with $Y_{a,b} = \delta_{a,p-b}$ and $A_k Y = A_{p-k-2}$. We now claim for $\mathbf{L} \in \mathbb{Z}_+^{p-3}$, $a, b = 1, \dots, p-1$, such that $a + b + L_1 + L_3 + \dots$ is even, that

$$(A_1^{L_1} A_2^{L_2} \dots A_{p-3}^{L_{p-3}})_{a,b} = \sum_{j=-\infty}^{\infty} \left\{ \binom{\mathbf{L}}{\frac{b-a}{2} + pj} - \binom{\mathbf{L}}{\frac{b+a}{2} + pj} \right\},$$

with on the right-hand side the supernomials of definition 2.1 (with $N = p - 3$).

Finally we note that the boson–fermion identities for the branching functions of the $A_1^{(1)}$ coset theories (5.21), which occur as a particular limit of the polynomial identities of theorem 5.1, can also be derived using the higher-level Bailey lemma of refs. [41, 42], as is the subject of ref. [18]. It is intriguing to note that though the method of ref. [18] and that of the present paper are very different, both rely on the use of the polynomial identities for the minimal models $M(p - k, p)$ [14, 17]. Whereas in ref. [18] they provide the necessary Bailey pairs as input for the higher-level lemma, in this paper they serve as initial conditions for the recursive proofs of the polynomial identities.

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Note added

We now have obtained a combinatorial interpretation of the boson-fermion identities of theorem 5.1 in terms of the inhomogeneous lattice paths introduced by Nakayashiki and Yamada [37] and Lascoux, Leclerc and Thibon [35].

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