# A NOTE ON THE TRINOMIAL ANALOGUE OF BAILEY'S LEMMA

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ABSTRACT. Recently, Andrews and Berkovich introduced a trinomial version of Bailey's lemma. In this note we show that each ordinary Bailey pair gives rise to a trinomial Bailey pair. This largely widens the applicability of the trinomial Bailey lemma and proves some of the identities proposed by Andrews and Berkovich.

**The trinomial Bailey lemma.** In a recent paper, Andrews and Berkovich (AB) proposed a trinomial analogue of Bailey's lemma [3]. As starting point AB take the following definitions of the $q$-trinomial coefficients

$$\binom{L;B;q}{A}_2=\sum_{j=0}^{\infty}\frac{q^{j(j+B)}(q)_L}{(q)_j(q)_{j+A}(q)_{L-2j-A}} \tag{1}$$

and

$$T_n(L,A,q)=q^{\frac{L(L-n)-A(A-n)}{2}}\binom{L;A-n;q^{-1}}{A}_2. \tag{2}$$

Here $(a)_\infty=\prod_{n=0}^{\infty}(1-aq^n)$ and $(a)_n=(a)_\infty/(aq^n)_\infty$, $n\in\mathbb{Z}$. To simplify equations it will also be convenient to introduce the notation

$$Q_n(L,A,q)=T_n(L,A,q)/(q)_L. \tag{3}$$

We note that the $q$-trinomial coefficients are non-zero for $-L\le A\le L$ only.

A pair of sequences $\tilde{\alpha}=\{\tilde{\alpha}_L\}_{L\ge 0}$ and $\tilde{\beta}=\{\tilde{\beta}_L\}_{L\ge 0}$ is said to form a trinomial Bailey pair relative to $n$ if

$$\tilde{\beta}_L=\sum_{r=0}^{L}Q_n(L,r,q)\,\tilde{\alpha}_r. \tag{4}$$

The trinomial analogue of the Bailey lemma is stated as follows [3].

**Lemma 1.** *If $(\tilde{\alpha},\tilde{\beta})$ is a trinomial Bailey pair relative to 0, then*

$$\sum_{L=0}^{\infty}(-1)_L\,q^{L/2}\tilde{\beta}_L=(-1)_{M+1}\sum_{L=0}^{\infty}\frac{\tilde{\alpha}_L}{q^{L/2}+q^{-L/2}}\,Q_1(M,L,q). \tag{5}$$

*Similarly, if $(\tilde{\alpha},\tilde{\beta})$ is a trinomial Bailey pair relative to 1, then*

$$\begin{aligned}\sum_{L=0}^{\infty}&\big(-q^{-1}\big)_L\,q^L\tilde{\beta}_L\\ &=(-1)_M\sum_{L=0}^{\infty}\tilde{\alpha}_L\Big\{Q_1(M,L,q)-\frac{Q_1(M-1,L+1,q)}{1+q^{-L-1}}-\frac{Q_1(M-1,L-1,q)}{1+q^{L-1}}\Big\}.\end{aligned} \tag{6}$$

As a corollary of their lemma, AB obtain the identities

$$\frac{1}{2}\sum_{L=0}^{\infty}(-1)_L\,q^{L/2}\tilde{\beta}_L=\frac{(-q)_\infty^2}{(q)_\infty^2}\sum_{L=0}^{\infty}\frac{\tilde{\alpha}_L}{q^{L/2}+q^{-L/2}}, \tag{7}$$

for a trinomial Bailey pair relative to 0, and

$$\frac{1}{2}\sum_{L=0}^{\infty}\bigl(-q^{-1}\bigr)_L\, q^L\tilde{\beta}_L=\frac{(-q)_\infty^2}{(q)_\infty^2}\sum_{L=0}^{\infty}\tilde{\alpha}_L\Bigl\{\frac{1}{1+q^{L+1}}-\frac{1}{1+q^{L-1}}\Bigr\}, \tag{8}$$

for a trinomial Bailey pair relative to 1.

**From binomial to trinomial Bailey pairs.** In ref. [3], the equations (7) and (8) are used to derive several new $q$-series identities. As input AB take trinomial Bailey pairs obtained from polynomial identities which on one side involve $q$-trinomial coefficients. Among these identities is an identity by the author which was stated in ref. [7] without proof, and therefore AB conclude "We have checked that his conjecture implies" followed by their equation (3.21), which is an identity for the characters of the $N=2$ superconformal models $SM(2p,(p-1)/2)$.

We now point out that equation (3.21) is a simple consequence of lemma 2 stated below. First we recall the definition of the ordinary (i.e., binomial) Bailey pair. A pair of sequences $(\alpha,\beta)$ such that

$$\beta_L=\sum_{r=0}^{L}\frac{\alpha_r}{(q)_{L-r}(aq)_{L+r}} \tag{9}$$

is said to form a Bailey pair relative to $a$.

**Lemma 2.** *Let $(\alpha,\beta)$ form a Bailey pair relative to $a=q^\ell$, where $\ell$ is a non-negative integer. For $n=0,1$, the following identity holds:*

$$\sum_{\substack{s=0\\ s\equiv L+\ell \ (\bmod\ 2)}}^{L-\ell}\frac{q^{s(s-n)/2}}{(q)_\ell(q)_s}\beta_{(L-s-\ell)/2}=\sum_{r=0}^{\infty}Q_n(L,2r+\ell,q)\,\alpha_r. \tag{10}$$

For $\ell>L$ the above of course trivializes to $0=0$.

Before proving lemma 2 we note an immediate consequence.

**Corollary 1.** *Let $(\alpha,\beta)$ form a Bailey pair relative to $a=q^\ell$ with non-negative integer $\ell$. Then $(\tilde{\alpha},\tilde{\beta})$ defined as*

$$\begin{gathered}\tilde{\alpha}_0,\dots,\tilde{\alpha}_{\ell-1}=0,\qquad \tilde{\alpha}_{2L+\ell}=\alpha_L,\qquad \tilde{\alpha}_{2L+\ell+1}=0,\qquad L\ge 0\\ \tilde{\beta}_0,\dots,\tilde{\beta}_{\ell-1}=0,\qquad \tilde{\beta}_{L+\ell}=\sum_{\substack{s=0\\ s\equiv L\ (\bmod\ 2)}}^{L}\frac{q^{s(s-n)/2}}{(q)_\ell(q)_s}\beta_{(L-s)/2},\qquad L\ge 0\end{gathered} \tag{11}$$

*forms a trinomial Bailey pair relative to $n=0,1$.*

*Proof.* The proof is trivial once one adopts the representation of the $q$-trinomial coefficients as given by equations (2.58) and (2.59) of ref. [2],

$$Q_n(L,A,q)=\frac{T_n(L,A,q)}{(q)_L}=\sum_{\substack{s=0\\ s\equiv L+A\ (\bmod\ 2)}}^{\infty}\frac{q^{s(s-n)/2}}{(q)_{\frac{L-A-s}{2}}(q)_{\frac{L+A-s}{2}}(q)_s}\,,\qquad n=0,1. \tag{12}$$

Now take the defining relation (9) of a Bailey pair with $a=q^\ell$ and make the replacement $L\to(L-s-\ell)/2$ where $s$ is an integer $0\le s\le L-\ell$ such that $s\equiv L+\ell \pmod 2$. After multiplication by $q^{s(s-n)/2}/(q)_s$ this becomes

$$\frac{q^{s(s-n)/2}}{(q)_s}\beta_{(L-s-\ell)/2}=(q)_\ell\sum_{r=0}^{\infty}\frac{\alpha_r q^{s(s-n)/2}}{(q)_{\frac{L-s-\ell}{2}-r}(q)_{\frac{L-s+\ell}{2}+r}(q)_s}. \tag{13}$$

Summing over $s$ yields equation (10). □

Returning to AB's paper, we note that their equation (3.21) simply follows from corollary 1 and the "$M(p-1,p)$ Bailey pairs" which arises from the $M(p-1,p)$ polynomial identities proven in refs. [4, 7]. Of course, an equivalent statement is that the "conjecture of ref. [7]", is proven using lemma 2 and the $M(p-1,p)$ Bailey pairs. To make this somewhat more explicit we consider the special case $p=3$. Then the $M(2,3)$ Bailey pairs are nothing but the entries A(1) and A(2) of Slater's list [6]. Specifically, A(1) contains the following Bailey pair relative to 1:

$$\alpha_L = \begin{cases} q^{6j^2-j}, & L=3j\geq 0 \\ q^{6j^2+j}, & L=3j>0 \\ -q^{6j^2-5j+1}, & L=3j-1>0 \\ -q^{6j^2+5j+1}, & L=3j+1>0 \end{cases} \qquad \text{and} \qquad \beta_L = \frac{1}{(q)_{2L}}. \tag{14}$$

By application of corollary 1 this gives the trinomial Bailey pair

$$\tilde{\alpha}_L = \begin{cases} q^{6j^2-j}, & L=6j\geq 0 \\ q^{6j^2+j}, & L=6j>0 \\ -q^{6j^2-5j+1}, & L=6j-1>0 \\ -q^{6j^2+5j+1}, & L=6j+1>0 \end{cases} \qquad \text{and} \qquad \tilde{\beta}_L = \sum_{\substack{s=0 \\ s\equiv L \ (\bmod\ 2)}}^{L} \frac{q^{s(s-n)/2}}{(q)_s(q)_{L-s}}. \tag{15}$$

Likewise, using entry A(2), we get

$$\tilde{\alpha}_L = \begin{cases} q^{6j^2-j}, & L=6j-1>0 \\ q^{6j^2+j}, & L=6j+1>0 \\ -q^{6j^2-5j+1}, & L=6j-3>0 \\ -q^{6j^2+5j+1}, & L=6j+3>0 \end{cases} \qquad \text{and} \qquad \tilde{\beta}_L = \sum_{\substack{s=0 \\ s\not\equiv L \ (\bmod\ 2)}}^{L} \frac{q^{s(s-n)/2}}{(q)_s(q)_{L-s}}. \tag{16}$$

Setting $n=0$ and summing up both trinomial Bailey pairs, we arrive at the trinomial Bailey pair of equations (3.18) and (3.19) of [3]. (Unlike the case $p\geq 4$, this trinomial Bailey pair was actually proven by AB, using theorem 5.1 of ref. [1].) Very similar results can be obtained through application of Slater's A(3) and A(4), A(5) and A(6), and A(7) and A(8).

**Conclusion.** We conclude this note with several remarks. First, it is of course not true that each trinomial Bailey pair is a consequence of an ordinary Bailey pair. The pairs given by equations (3.13) and (3.14) of ref. [3] being examples of irreducible trinomial Bailey pairs. Second, if one replaces $Q_n(L,r,q)$ by its $q$-multinomial analogue [5, 8] and takes that as the definition of a $q$-multinomial Bailey pair, it becomes straightforward to again construct multinomial Bailey pairs out of ordinary ones. Finally, it is worthwhile to note that the Bailey flow from the minimal model $M(p,p+1)$ to the $N=2$ superconformal model $SM(2p,(p-1)/2)$ as concluded by AB could now be replaced by $M(p-1,p)\to N=2\ SM(2p,(p-1)/2)$. Perhaps better though would be to write $M(p-1,p)\to M(p,p+1)\to N=2\ SM(2p,(p-1)/2)$, where the first arrow indicates the flow induced by corollary 1 and the second arrow the flow induced by (7) and (8).

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