On the q-analogue of the sum of cubes

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Abstract

A simple q-analogue of the sum of cubes is given. This answers a question posed in this journal by Garrett and Hummel.

The sum of cubes and its q-analogues

It is well-known that the first n consecutive cubes can be summed in closed form as

$$\sum_{k=1}^{n} k^3 = \binom{n+1}{2}^2.$$

Recently, Garrett and Hummel discovered the following q-analogue of this result:

$$\sum_{k=1}^{n} q^{k-1} \frac{(1-q^k)^2(2-q^{k-1}-q^{k+1})}{(1-q)^2(1-q^2)} = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}^2, \tag{1}$$

where

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(1-q^{n-k+1})(1-q^{n-k+2})\cdots(1-q^n)}{(1-q)(1-q^2)\cdots(1-q^k)}$$

is a q-binomial coefficient.

In their paper Garrett and Hummel commiserate the fact that (1) is not as simple as one might have hoped, and ask for a simpler sum of q-cubes. In response to this I propose the identity

$$\sum_{k=1}^{n} q^{2n-2k} \frac{(1-q^k)^2(1-q^{2k})}{(1-q)^2(1-q^2)} = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}^2. \tag{2}$$

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Proof. Since
$$\begin{bmatrix} n+1 \\ 2 \end{bmatrix}^2 - q^2 \begin{bmatrix} n \\ 2 \end{bmatrix}^2 = \frac{(1-q^n)^2(1-q^{2n})}{(1-q)^2(1-q^2)}$$
equation (2) immediately follows by induction on n. $\square$

The form of (2) should not really come as a surprise in view of the fact that the q-analogue of the sum of squares

$$\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$$

is given by

$$\sum_{k=1}^{n} q^{2n-2k} \frac{(1-q^k)(1-q^{3k})}{(1-q)(1-q^3)} = \frac{(1-q^n)(1-q^{n+1})(1-q^{2n+1})}{(1-q)(1-q^2)(1-q^3)},$$

and the q-analogue of
$$\sum_{k=1}^{n} k = \binom{n+1}{2}$$
is
$$\sum_{k=1}^{n} q^{2n-2k} \frac{(1-q^k)}{(1-q)} = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}.$$

References

[1] K. C. Garrett and K. Hummel, *A combinatorial proof of the sum of q-cubes*, Electron. J. Combin. **11** (2004), R9, 6pp.