A Selberg integral for the root system of type A

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Motivation

Some early history …

- Wallis integral (1656)

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = \frac{\pi}{4}.$$

- Gamma function (Euler 1720s)

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} \mathrm{e}^{-t} \mathrm{d}t$$

for $\mathrm{Re}(\alpha) > 0$.

Replacing $x^2 = t$, Euler observed that Wallis' integral may be written as

$$\int_0^1 t^{1/2-1}(1-t)^{3/2-1}\mathrm{d}t = \Gamma(1/2)\Gamma(3/2).$$

This led Euler to the discovery of a more general integral.

- Euler beta integral (1730s)

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\mathrm{d}t = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

for $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$.

Note that if we replace $\beta \to \zeta \in \mathbb{R}$ and $t \to t/\zeta$ and let $\zeta \to \infty$ this returns the integral representation of the gamma function.

Some recent history . . .

- Selberg integral (1944)

$$\int_{[0,1]^n} \prod_{i=1}^{n} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1\le i<j\le n} |t_i - t_j|^{2\gamma} \mathrm{d}t$$
$$= n! \prod_{i=0}^{n-1} \frac{\Gamma(\alpha+i\gamma)\Gamma(\beta+i\gamma)\Gamma(\gamma+i\gamma)}{\Gamma(\alpha+\beta+(n+i-1)\gamma)\Gamma(\gamma)}$$

for $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > \cdots$.

- Classical viewpoint: The Selberg integral is associated to the A_{n-1} root system

  $$\Phi = \{\pm(\epsilon_i - \epsilon_j) | 1 \le i < j \le n\}$$

  with ϵ_i the ith standard unit vector in $\mathbb{R}^n$.

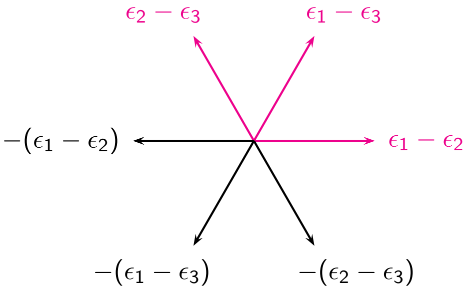
  The Vandermonde product

  $$\Delta(t) = \prod_{1 \le i < j \le n} (t_i - t_j)$$

  occurring in the Selberg integral may be viewed as a formal product over set of positive roots Φ_+ (i.e., roots of the form $\epsilon_i - \epsilon_j$)

  $$\Delta(t) \sim \prod_{\alpha \in \Phi_+} \left(1 - \exp(\alpha)\right)$$

  with $t_i = \exp(\epsilon_i)$.



The root system A_2 with positive roots in pink.

- Alternative viewpoint: Selberg integral associated to $\mathfrak{sl}_2$ (or A_1).

  $V = V_1 \otimes \cdots \otimes V_n$ tensor product of $\mathfrak{sl}_2$ modules.

  Knizhnik–Zamolodchikov (KZ) equation for V-valued function $u(x_1, \dots, x_n)$.

  Selberg integral arises as "coordinate function" of hypergeometric solution to KZ equation with values in

  $$\{v \in L_\ell \otimes L_m | hv = (\ell + m - 2n)v,\ ev = 0\}$$

  with L_ℓ an irreducible $\mathfrak{sl}_2$ highest weight module of weight ℓ, and $e,\ h$ the generators of the Borel subalgebra of $\mathfrak{sl}_2$.

A_n Selberg Integral

Adopting the alternative point of view and labelling the Selberg integral by $\mathfrak{sl}_2$ or A_1, we have to ask:

$$\boxed{\text{What/where is the } \mathfrak{sl}_{n+1} \text{ or } \mathrm{A}_n \text{ Selberg integral?}}$$

Let

$$\{a_i = \epsilon_i - \epsilon_{i+1} \,|1 \le i \le n\}$$

be the set of simple roots of A_n.

Then the Cartan matrix of A_n is given by

$$\left(a_i \cdot a_j\right)_{1\le i,j\le n} = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & & \ddots & \\ & & \ddots & & -1 \\ & & & -1 & 2 \end{pmatrix}$$

To each simple root a_s attach a set of variables

$$t^{(s)} = (t_1^{(s)}, \dots, t_{k_s}^{(s)})$$

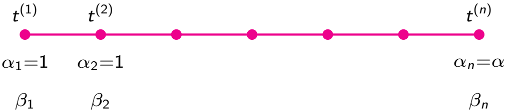
such that $0 \le k_1 \le k_2 \le \cdots \le k_n$.

Set $k_0 = k_{n+1} = 0$ and let $\alpha, \beta_1, \dots, \beta_n, \gamma \in \mathbb{C}$ subject to several mild restrictions, such as

$$\mathrm{Re}(\alpha) > 0,\ \mathrm{Re}(\beta_1) > 0, \dots, \mathrm{Re}(\beta_n) > 0.$$

Set

$$(\alpha_1, \dots, \alpha_{n-1}, \alpha_n) = (1, \dots, 1, \alpha).$$



Define the generalised Vandermonde product

$$\Delta(u,v) = \prod_{i,j\geq 1}(u_i - v_j)$$

and let

$$C^{k_1,\dots,k_n}[0,1] \subset [0,1]^{k_1+\dots+k_n}$$

be an appropriate integration domain, too complicated to describe in a 25 minute talk.

Theorem (A$_n$ Selberg integral)

$$\int_{C^{k_1,\dots,k_n}[0,1]} \prod_{i=1}^{k_n} \bigl(t_i^{(n)}\bigr)^{\alpha-1} \prod_{s=1}^{n} \prod_{i=1}^{k_s} \bigl(1-t_i^{(s)}\bigr)^{\beta_s-1} \times \prod_{s=1}^{n-1} \bigl|\Delta\bigl(t^{(s)},t^{(s+1)}\bigr)\bigr|^{-\gamma} \prod_{s=1}^{n} \bigl|\Delta\bigl(t^{(s)}\bigr)\bigr|^{2\gamma}\, \mathrm{d}t^{(1)}\cdots\mathrm{d}t^{(n)}$$

$$= \prod_{1\le s\le r\le n} \prod_{i=1}^{k_s-k_{s-1}} \frac{\Gamma(\beta_s+\dots+\beta_r+(i+s-r-1)\gamma)}{\Gamma(\alpha_r+\beta_s+\dots+\beta_r+(i+s-r+k_r-k_{r+1}-2)\gamma)} \times \prod_{s=1}^{n} \prod_{i=1}^{k_s} \frac{\Gamma(\alpha_s+(i-k_{s+1}-1)\gamma)\Gamma(i\gamma)}{\Gamma(\gamma)}.$$

q-Binomial Theorem I

For $k \in \mathbb{N}$ and $z \in \mathbb{C}$ the q-Pochhammer symbols are

$$(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1})$$

$$(a;q)_\infty = (1-a)(1-aq)\cdots$$

and

$$(a;q)_z = \frac{(a;q)_\infty}{(aq^z;q)_\infty}.$$

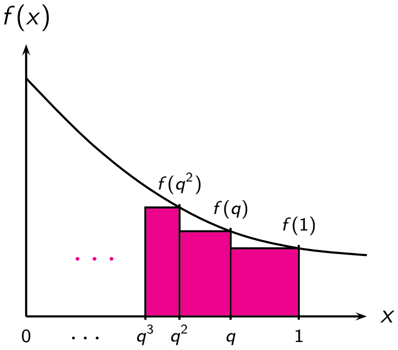
Then the q-binomial theorem is given by

$$\sum_{k=0}^{\infty} \frac{(b;q)_k}{(q;q)_k} z^k = \frac{(bz;q)_\infty}{(z;q)_\infty}.$$

Let

$$\int_0^1 f(x)\mathrm{d}_q x = (1-q)\sum_{i=0}^{\infty} f(q^i)q^i$$

be the Jackson or q-integral.



Then the q-binomial theorem with $z = q^{\alpha}$ and $b = q^{\beta}$ may be written as the q-beta integral

$$\int_0^1 t^{\alpha-1}(tq;q)_{\beta-1}\,\mathrm{d}_q t = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)},$$

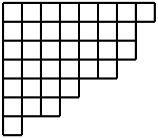
where Γ_q is the q-gamma function.

In the $q \to 1^-$ limit the q-binomial theorem thus yields the Euler beta integral.

Macdonald Polynomials

Let x be an n-letter alphabet with letters $x_1, x_2, \ldots, x_n$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition

$\lambda =$



The q-shift operator T_{q,x_i} is defined as

$$T_{q,x_i}\big(f(x)\big) = f(x_1, \ldots, x_{i-1}, qx_i, x_{i+1}, \ldots, x_n).$$

and Macdonald's commuting family $\{D_r\}_{r=0}^{n}$ of q-difference operators is given by

$$D_r = t^{\binom{r}{2}} \sum_{\substack{I \subseteq [n] \\ |I|=r}} \prod_{\substack{i \in I \\ j \notin I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{i \in I} T_{q,x_i}.$$

Defining the generating series of the D_r as

$$D(u;q,t)=\sum_{r=0}^{n} D_r u^r$$

the Macdonald polynomials $P_\lambda(x;q,t)$ are the eigenfunctions of $D(u;q,t)$ with eigenvalue

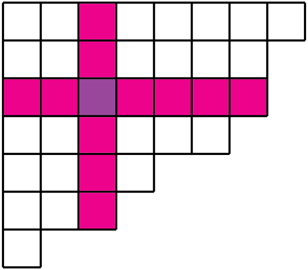
$$\prod_{i=1}^{n}(1+ut^{n-i}q^{\lambda_i}).$$

For $q=t$ the Macdonald polynomials simplify to the well-known Schur functions

$$P_\lambda(x;t,t)=s_\lambda(x)=\frac{\det_{1\le i,j\le n}(x_i^{\lambda_j+n-j})}{\det_{1\le i,j\le n}(x_i^{n-j})}.$$

Cauchy Identity

Given a partition λ, each of its squares s is assigned four integers, known as the arm-length $a(s)$, leg-length $l(s)$, arm-colength $a'(s)$ and leg-colength $l'(s)$.



The arm-length of 🟪 is 4. The leg-length of 🟪 is 3. The arm- and leg-colengths of 🟪 are both 2.

The Cauchy identity for Macdonald polynomials is

$$\sum_{\lambda} P_\lambda(x;q,t) P_\lambda(y;q,t) \prod_{s\in\lambda} \frac{1-q^{a(s)}t^{l(s)+1}}{1-q^{a(s)+1}t^{l(s)}} = \prod_{i,j\geq 1} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}.$$

When $q=t$ this reduces to the well-known Cauchy determinant

$$\det_{1\leq i\leq j\leq n}\left(\frac{1}{1-x_iy_j}\right) = \frac{\Delta(x)\Delta(y)}{\prod_{i,j=1}^{n}(1-x_iy_j)}.$$

q-Binomial Theorem II

The power sums p_r are given by $p_0 = 1$ and

$$p_r(x) = \sum_{i \geq 1} x_i^r .$$

The map $\epsilon_{b,t}$ — acting on symmetric functions of y — is defined by its action on the p_r:

$$\epsilon_{b,t}\big(p_r(y)\big) = \frac{1 - b^r}{1 - t^r}.$$

A theorem of Macdonald states that

$$\epsilon_{b,t}\big(P_\lambda(y; q, t)\big) = \prod_{s \in \lambda} \frac{t^{l'(s)} - b\, q^{a'(s)}}{1 - q^{a(s)}\, t^{l(s)+1}}.$$

It may also be shown that

$$\epsilon_{b,t}\left(\prod_{i,j\geq 1}\frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}\right)=\prod_{i\geq 1}\frac{(b\,x_i;q)_\infty}{(x_i;q)_\infty}.$$

Applying the map $\epsilon_{b,t}$ to the Cauchy identity we thus obtain an n-dimensional analogue of the q-binomial theorem:

$$\boxed{\sum_\lambda P_\lambda(x;q,t)\prod_{s\in\lambda}\frac{t^{l'(s)}-b\,q^{a'(s)}}{1-q^{a(s)+1}t^{l(s)}}=\prod_{i=1}^{n}\frac{(b\,x_i;q)_\infty}{(x_i;q)_\infty}}$$

If $n=1$ then $x=(x_1)$, $\lambda=(k)$ and

$$P_{(k)}(x;q,t)=x_1^k, \qquad \prod_{s\in\lambda}\frac{t^{l'(s)}-b\,q^{a'(s)}}{1-q^{a(s)+1}t^{l(s)}}=\frac{(b;q)_k}{(q;q)_k}$$

so that we recover the classical q-binomial theorem (with $z\to x_1$).

Taking

$$\begin{aligned} x_i &= q^{\alpha+\gamma(n-i)} \quad \text{for} \quad 1 \leq i \leq n \\ t &= q^{\gamma} \\ b &= q^{\beta} \end{aligned}$$

in the n-dimensional q-binomial theorem yields an n-dimensional q-integral, generalising the q-beta integral.

In the $q \to 1^-$ limit this gives the Selberg integral.

To prove the A_n Selberg integral we need a further generalisation of the q-binomial theorem!

q-Binomial Theorem III

In 3 more hours I could have described a q-binomial theorem of the form

$$\sum_{\lambda^{(1)},\dots,\lambda^{(n)}} P_{\lambda^{(1)}}(x^{(1)};q,t)\cdots P_{\lambda^{(n)}}(x^{(n)};q,t) \\ \times \big(\text{stuff with arms and legs}\big) = \text{infinite product}$$

with $x^{(s)}=(x_1^{(s)},\dots,x_{k_s}^{(s)})$ and $k_1\le k_2\le\cdots\le k_n$.

$\Rightarrow$ A $(k_1+\cdots+k_n)$-dimensional q-integral

$\Rightarrow$ The A_n Selberg integral.

Open Problems

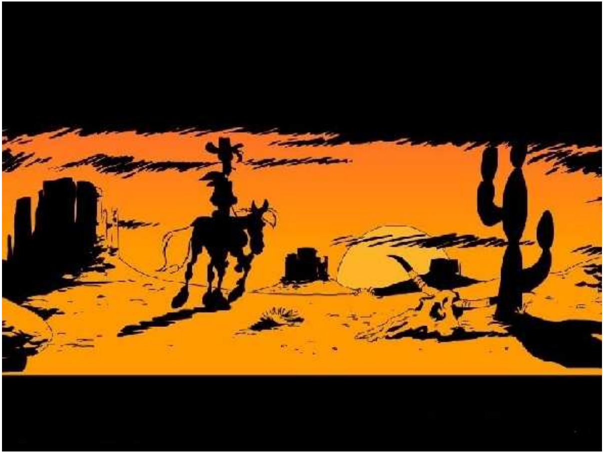
- Can we evaluate the more general integral

$$\int_{C^{k_1,\dots,k_n}[0,1]} \prod_{s=1}^{n}\prod_{i=1}^{k_s}\bigl(t_i^{(s)}\bigr)^{\alpha_s-1}\bigl(1-t_i^{(s)}\bigr)^{\beta_s-1} \\ \times \prod_{s=1}^{n-1}\bigl|\Delta\bigl(t^{(s)},t^{(s+1)}\bigr)\bigr|^{-\gamma}\prod_{s=1}^{n}\bigl|\Delta\bigl(t^{(s)}\bigr)\bigr|^{2\gamma}\,\mathrm{d}t^{(1)}\cdots\mathrm{d}t^{(n)}\ ?$$

- Can we remove the ordering

$$0\le k_1\le k_2\le\cdots\le k_n\ ?$$

- Can we generalise to other root systems ?



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