Bisymmetric functions, Macdonald polynomials and $\mathfrak{sl}_3$ basic hypergeometric series

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Motivation

- Euler beta integral (17??)

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\mathrm{d}t = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

for $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$.

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- Selberg integral (1944)

$$\int_{[0,1]^n} \prod_{i=1}^{n} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1\le i<j\le n} |t_i - t_j|^{2\gamma}\mathrm{d}t = n! \prod_{i=0}^{n-1} \frac{\Gamma(\alpha+i\gamma)\Gamma(\beta+i\gamma)\Gamma(\gamma+i\gamma)}{\Gamma(\alpha+\beta+(n+i-1)\gamma)\Gamma(\gamma)}$$

for $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\gamma) > \cdots$.

- Classical viewpoint: Selberg integral associated to the A_{n-1} root system

$$\Phi = \{\pm(\epsilon_i - \epsilon_j) | 1 \leq i < j \leq n\}$$

with ϵ_i the ith standard unit vector in $\mathbb{R}^n$.

$$\Delta_n(t) = \prod_{1 \leq i < j \leq n} |t_i - t_j|^2 \sim \prod_{\alpha \in \Phi} |1 - \exp(\alpha)|$$

with $t_i = \exp(\epsilon_i)$, and

$$n! = \mathrm{order}(W_{\mathrm{A}_{n-1}})$$

with $W_{\mathrm{A}_{n-1}} \simeq S_n$ the A_{n-1} Weyl group.

- Alternative viewpoint: Selberg integral associated to $\mathfrak{sl}_2$ (or A_1).

  $V = V_1 \otimes \cdots \otimes V_n$ tensor product of $\mathfrak{sl}_2$ modules.

  Knizhnik–Zamolodchikov (KZ) equation for V-valued function $u(x_1, \dots, x_n)$.

  Selberg integral arises as "coordinate function" of hypergeometric solution to KZ equation with values in

$$\{v \in L_\ell \otimes L_m | hv = (\ell + m - 2n)v, \; ev = 0\}$$

  with L_ℓ an irreducible $\mathfrak{sl}_2$ highest weight module of weight ℓ, and $e, \; h$ the generators of the Borel subalgebra of $\mathfrak{sl}_2$.

- Tarasov & Varchenko (2003)

$$\int_{C_{m,n;\gamma}} \prod_{i=1}^{n} t_i^{\alpha-1}(1-t_i)^{\beta_1-1} \prod_{j=1}^{m}(1-s_j)^{\beta_2-1} \prod_{i=1}^{n}\prod_{j=1}^{m} |t_i - s_j|^{-\gamma}$$

$$\times \Delta_n^{\gamma}(t)\, \Delta_m^{\gamma}(s)\, w(t;s;0)\, \mathrm{d}t\, \mathrm{d}s$$

$$= n!m! \prod_{i=0}^{n-1} \frac{\Gamma(\alpha+i\gamma)\Gamma(\gamma+i\gamma)}{\Gamma(\gamma)}$$

$$\times \prod_{i=0}^{n+m-1} \frac{\Gamma(\beta_1+i\gamma)}{\Gamma(\alpha+\beta_1+(n-2m+i-1)\gamma)}$$

$$\times \prod_{i=0}^{m-1} \frac{\Gamma(\beta_2+i\gamma)\Gamma(\gamma+i\gamma)}{\Gamma(\beta_2+(2n-2m+i-1)\gamma)\Gamma(\gamma)}$$

$$\times \frac{\Gamma(\beta_1+\beta_2-\gamma-1+i\gamma)}{\Gamma(\alpha+\beta_1+\beta_2-1+(n+i-2)\gamma)}$$

Here $t=(t_1,\dots,t_n)$, $s=(s_1,\dots,s_m)$, $w(t;s;\gamma)$ a bisymmetric function and $C_{m,n;\gamma} \in \mathbb{R}^{n+m}$ a complicated domain of integration.

The Tarasov–Varchenko integral may be viewed as an $\mathfrak{sl}_3$ generalization of the Selberg integral (recovered for $m = 0$) and arises as "coordinate function" of the hypergeometric solution of the $\mathfrak{sl}_3$ KZ equation in some special subspace (labelled by n and m) of the full parameter space.

The Tarasov–Varchenko integral may be viewed as an $\mathfrak{sl}_3$ generalization of the Selberg integral (recovered for $m = 0$) and arises as "coordinate function" of the hypergeometric solution of the $\mathfrak{sl}_3$ KZ equation in some special subspace (labelled by n and m) of the full parameter space.

$m = n = 1$: $\mathfrak{sl}_3$ beta integral

$$\int_{0\le t\le s\le 1} t^{\alpha-1}(1-t)^{\beta_1-1}(1-s)^{\beta_2-1}(s-t)^{-\gamma-1}\,\mathrm{d}t\,\mathrm{d}s$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_1+\gamma)}{\Gamma(\alpha+\beta_1-2\gamma)\Gamma(\alpha+\beta_1-\gamma)\Gamma(\beta_2-\gamma)}$$

$$\times \frac{\Gamma(\beta_1+\beta_2-\gamma-1)}{\Gamma(\alpha+\beta_1+\beta_2-1-\gamma)}$$

Paradigm

Hypergeometric integrals $\Leftrightarrow$ Hypergeometric series

There should "exist" $\mathfrak{sl}_3$ hypergeometric and basic hypergeometric series of the form

$$\sum \text{hypergeometric terms} \times \text{bisymmetric function.}$$

Bisymmetric functions

Notation

- $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ with $0 \leq m \leq n$
- $\Delta(x; t) = \prod_{1 \leq i \leq j \leq n} \frac{1 - t^{-1} x_i / x_j}{1 - x_i / x_j}$
- $\Delta(y; t) = \prod_{1 \leq i \leq j \leq m} \frac{1 - t^{-1} y_i / y_j}{1 - y_i / y_j}$

The bisymmetric function ω

$$\omega(x,y;t) = \frac{(1-t^{-1})^{n+m}}{(t^{-1};t^{-1})_{n-m}(t^{-1};t^{-1})_m} \times \sum_{w\in S_n\times S_m} w\Bigg(\Delta(x;t)\Delta(y;t) \times \prod_{i=1}^{m} \frac{1}{1-t^{-1}y_i/x_{i+n-m}} \times \prod_{1\le i<j\le m} \frac{1-y_i/x_{j+n-m}}{1-t^{-1}y_i/x_{j+n-m}}\Bigg)$$

- Hypergeometric limit:

$$\lim_{q\to 1} \omega(q^x, q^y; q^\gamma) = \frac{(-\gamma)^m n!}{(n-m)!} w(x, y; \gamma)$$

with $w(x, y; \gamma)$ the bisymmetric function of the $\mathfrak{sl}_3$ Selberg integral.

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- Homogeneity: $\omega(ax, ay; t) = \omega(x, y; t)$

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- Initial condition: $\omega(x, -; t) = 1$

- **Recursion:** For $x^{(l)} = (x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_n)$ and $y^{(k)} = (y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_m)$

$$\omega(x,y;t) = \sum_{l=1}^{n} \omega(x^{(l)}, y^{(k)}; t)\, \frac{1-t^{-1}}{1-t^{-1}y_k/x_l} \times \prod_{\substack{i=1\\ i\neq k}}^{m} \frac{1-y_i/x_l}{1-t^{-1}y_i/x_l} \prod_{\substack{i=1\\ i\neq l}}^{n} \frac{1-t^{-1}x_i/x_l}{1-x_i/x_l}$$

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$$\omega(x, y; t) = \sum_{l=1}^{n} \omega(x^{(l)}, y^{(k)}; t) \frac{1 - t^{-1}}{1 - t^{-1} y_k / x_l} \times \prod_{\substack{i=1 \\ i \neq k}}^{m} \frac{1 - y_i / x_l}{1 - t^{-1} y_i / x_l} \prod_{\substack{i=1 \\ i \neq l}}^{n} \frac{1 - t^{-1} x_i / x_l}{1 - x_i / x_l}$$

- Stability: $\omega(x, y; t)|_{x_i = y_j} = \omega(x^{(i)}, y^{(j)}; t)$
  for $1 \leq i \leq n$ and $1 \leq j \leq m$

- Recursion: For $x^{(l)} = (x_1, \dots, x_{l-1}, x_{l+1}, \dots, x_n)$ and $y^{(k)} = (y_1, \dots, y_{k-1}, y_{k+1}, \dots, y_m)$

$$\omega(x,y;t) = \sum_{l=1}^{n} \omega(x^{(l)}, y^{(k)}; t) \, \frac{1-t^{-1}}{1-t^{-1}y_k/x_l} \times \prod_{\substack{i=1 \\ i\neq k}}^{m} \frac{1-y_i/x_l}{1-t^{-1}y_i/x_l} \prod_{\substack{i=1 \\ i\neq l}}^{n} \frac{1-t^{-1}x_i/x_l}{1-x_i/x_l}$$

- Stability: $\omega(x,y;t)|_{x_i=y_j} = \omega(x^{(i)}, y^{(j)}; t)$ for $1 \le i \le n$ and $1 \le j \le m$

- Principal specialization:

$$\omega((1,t,\dots,t^{n-1}),y;t) = \prod_{i=1}^{m} \frac{1-t^{i-n-1}}{1-y_i t^{-n}}$$

- Alternating sign matrices: For $m = n$

$$\omega(x,y;t) = \frac{(1-t^{-1})^n}{\prod_{i,j=1}^n (1-t^{-1}y_i/x_j)}$$

$$\times \sum_A (1-t)^{2N(A)} t^{\binom{n}{2}-\mathcal{I}(A)} \prod_{i=1}^n x_i^{N_i(A)} y_i^{N^i(A)} \prod_{\substack{i,j=1\\ a_{ij}=0}}^n (\alpha_{ij}x_i - y_j)$$

- A: $n \times n$ alternating sign matrix with entries $a_{ij} \in \{-1,0,1\}$
- $N_i(A)$: # of -1s in row i
- $N^i(A)$: # of -1s in column i
- $N(A) = \sum_i N_i(A) = \sum_i N^i(A)$
- $\mathcal{I}(A)$: inversion number

$$\mathcal{I}(A) = \sum_{1\le i<i'\le n} \sum_{1\le j<j'\le n} a_{ij}a_{i'j'}$$

- $\alpha_{ij} = \begin{cases} t & \text{if } \sum_{k=1}^{j} a_{ik} = \sum_{k=1}^{i} a_{kj} \\ 1 & \text{otherwise} \end{cases}$

- Modified Cauchy identity (Conjecture)

  For $m = n$

$$\sum_{\lambda} b_\lambda(q,t) P_\lambda(x;q,t) P_\lambda(y;q,t) \prod_{i=1}^{n} (1 - q^{\lambda_i} t^{n-i+1})$$

$$= \omega(x, y^{-1}; t) \prod_{i=1}^{n} \frac{1}{x_i y_i} \prod_{i,j=1}^{n} \frac{(t x_i y_j; q)_\infty}{(x_i y_j; q)_\infty}$$

  - $P_\lambda(x;q,t)$ Macdonald polynomial labelled by the partition λ
  - $b_\lambda(q,t) = \dfrac{c_\lambda(q,t)}{c'_\lambda(q,t)}$ with

$$c_\lambda(q,t) = \prod_{s\in\lambda} (1 - q^{a(s)} t^{l(s)+1})$$

$$c'_\lambda(q,t) = \prod_{s\in\lambda} (1 - q^{a(s)+1} t^{l(s)})$$

  - Cauchy identity:

$$\sum_{\lambda} b_\lambda(q,t) P_\lambda(x;q,t) P_\lambda(y;q,t) = \prod_{i,j=1}^{n} \frac{(t x_i y_j; q)_\infty}{(x_i y_j; q)_\infty}$$

$\mathfrak{sl}_2$ Basic hypergeometric series

Classical (one-variable) BHS

$$ {}_{r+1}\phi_r \left[\begin{matrix} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix} ; q, z \right] = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_{r+1}; q)_k}{(q, b_1, \ldots, b_r; q)_k} \, z^k $$

Condensed notation:

$$ (a_1, \ldots, a_k; q)_k = (a_1; q)_k \cdots (a_k; q)_k $$

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Condensed notation:

$$ (a_1, \ldots, a_k; q)_k = (a_1; q)_k \cdots (a_k; q)_k $$

q-Binomial theorem:

$$ {}_1\phi_0\left[\begin{matrix} a \\ - \end{matrix}; q, z\right] = \frac{(az; q)_\infty}{(z; q)_\infty} $$

multivariable BHS of $\mathfrak{sl}_2$ type

Kaneko and Macdonald (independently):

$$
{}_{r+1}\Phi_r\left[\begin{matrix} a_1, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, t, x\right] = \sum_{\lambda} t^{n(\lambda)} \frac{P_\lambda(x; q, t)}{c'_\lambda(q, t)} \frac{(a_1, \dots, a_{r+1}; q, t)_\lambda}{(b_1, \dots, b_r; q, t)_\lambda}
$$

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$$

- $x=(x_1,\ldots,x_n)$
- $P_\lambda(x;q,t)$ Macdonald polynomial labelled by the partition λ
  - $n=1$: $P_{(k)}(z;q,t)=z^k$
- $n(\lambda)=\sum_{i\geq 1}(i-1)\lambda_i$

- $c'_\lambda(q,t) = \prod_{s\in\lambda}(1-q^{a(s)+1}t^{l(s)})$
  - $n=1$: $c'_{(k)}(q,t) = (q;q)_k$
- $(b;q,t)_\lambda = \prod_{s\in\lambda}\left(1-bq^{a'(s)}t^{-l'(s)}\right) = \prod_{i\geq 1}(bt^{1-i};q)_{\lambda_i}$
  - $n=1$: $(b;q,t)_{(k)} = (b;q)_k$
- Condensed notation:

$$(a_1,\ldots,a_k;q,t)_\lambda = (a_1;q,t)_\lambda\cdots(a_k;q,t)_\lambda$$

- $n=1$:

$${}_{r+1}\Phi_r\left[\begin{matrix}a_1,\ldots,a_{r+1}\\ b_1,\ldots,b_r\end{matrix};q,t,(z)\right] = {}_{r+1}\phi_r\left[\begin{matrix}a_1,\ldots,a_{r+1}\\ b_1,\ldots,b_r\end{matrix};q,z\right]$$

$\mathfrak{sl}_2$ q-binomial theorem: (Kaneko and Macdonald)

$$ {}_1\Phi_0\left[\begin{matrix} a \\ - \end{matrix}; q, t, x\right] = \prod_{i=1}^{n} \frac{(ax_i; q)_\infty}{(x_i; q)_\infty} $$

$\mathfrak{sl}_3$ Basic hypergeometric series

multivariable BHS of $\mathfrak{sl}_3$ type

$$
\begin{aligned}
&{}_{r+1}\Phi_r\left[\begin{matrix} a_1,\dots,a_{r+1} \\ b_1,\dots,b_r \end{matrix}; q,t,x,y\right] \\
&= \prod_{i=1}^{m} \frac{(y_i;q)_\infty}{(y_i t^{m-n-1};q)_\infty} \\
&\quad \times \sum_{\lambda,\mu} t^{n(\lambda)+n(\mu)} \frac{P_\mu(x;q,t)}{c'_\mu(q,t)} \frac{P_\lambda(y;q,t)}{c'_\lambda(q,t)} \\
&\qquad \times (qt^{m-1};q,t)_\lambda \, W^{(n-m)}_{\mu\lambda}(q,t) \\
&\qquad \times \frac{(a_1,\dots,a_{r+1};q,t)_\mu}{(b_1,\dots,b_r;q,t)_\mu} \\
&\qquad \times \prod_{i=1}^{m}\prod_{j=1}^{n} \frac{(qt^{m-n+j-i-1};q)_{\lambda_i-\mu_j}}{(qt^{m-n+j-i};q)_{\lambda_i-\mu_j}}
\end{aligned}
$$

- $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_m)$ with $0 \le m \le n$
- Let

$$q^{\mu} t^{\delta^{(n)}} = (q^{\mu_1} t^{n-1}, q^{\mu_2} t^{n-2}, \ldots, q^{\mu_n})$$

and

$$q^{\lambda} t^{\delta^{(m)}} = (q^{\lambda_1} t^{m-1}, q^{\lambda_2} t^{m-2}, \ldots, q^{\lambda_m}).$$

Then

$$W_{\mu\lambda}^{(n-m)}(q,t) = \omega(q^{\mu} t^{\delta^{(n)}}, q^{\lambda} t^{\delta^{(m)}}; t).$$

- $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_m)$ with $0 \leq m \leq n$
- Let
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and
$$q^{\lambda} t^{\delta^{(m)}} = (q^{\lambda_1} t^{m-1}, q^{\lambda_2} t^{m-2}, \ldots, q^{\lambda_m}).$$
Then
$$W_{\mu\lambda}^{(n-m)}(q,t) = \omega(q^{\mu} t^{\delta^{(n)}}, q^{\lambda} t^{\delta^{(m)}}; t).$$
    - $W^{(0)}_{(3,2,1),(4,1,1)}(q,t) = W^{(0)}_{(3,2,1,0),(4,1,1,0)}(q,t)$
    - $W^{(0)}_{(3,2,1),(4,1,1)}(q,t) \neq W^{(1)}_{(3,2,1,0),(4,1,1)}(q,t)$

Results and conjectures

- (Simple)

$$ {}_{r+1}\Phi_r\left[\begin{matrix} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix}; q, t, x, -\right] = {}_{r+1}\Phi_r\left[\begin{matrix} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix}; q, t, x\right] $$

Results and conjectures

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$$ {}_{r+1}\Phi_r \left[\begin{matrix} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix} ; q, t, x, - \right] = {}_{r+1}\Phi_r \left[\begin{matrix} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix} ; q, t, x \right] $$

- (Moderately simple)

$$ {}_{r+1}\Phi_r \left[\begin{matrix} 1, a_2, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix} ; q, t, x, y \right] = 1 $$

Results and conjectures

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- (Moderately simple)

$$ {}_{r+1}\Phi_r\left[\begin{matrix} 1,a_2,\ldots,a_{r+1} \\ b_1,\ldots,b_r \end{matrix}; q,t,x,y\right] = 1 $$

- (Conjecture; True if modified Cauchy identity is true; Hard)

$$ {}_{r+1}\Phi_r\left[\begin{matrix} a_1,\ldots,a_{r+1} \\ b_1,\ldots,b_r \end{matrix}; q,t,x,y\right] = {}_{r+1}\Phi_r\left[\begin{matrix} a_1,\ldots,a_{r+1} \\ b_1,\ldots,b_r \end{matrix}; q,t,y,x\right] $$

for $m = n$.

- (Conjecture) $\mathfrak{sl}_3$ q-binomial theorem I

$$ {}_1\Phi_0\left[\begin{matrix} a \\ - \end{matrix}; q,t,x,y\right] = \prod_{i=1}^{m} \frac{(azy_i t^{m-1};q)_\infty}{(zy_i t^{m-1};q)_\infty} \prod_{i=1}^{n-m} \frac{(azt^{m+i-1};q)_\infty}{(zt^{m+i-1};q)_\infty} $$

for $x = (zt^{n-1}, zt^{n-2}, \dots, zt, z)$

  - True for $n = m = 1$ (easy)
  - True for $n = 2,\ m = 1$ (hard)
  - True for $n = m = 2$ (hard)

- (Conjecture) $\mathfrak{sl}_3$ q-binomial theorem II

$$ {}_1\Phi_0\left[\begin{matrix} a \\ - \end{matrix}; q,t,x,y\right] = \prod_{i=1}^{n} \frac{(azx_i t^{n-1};q)_\infty}{(zx_i t^{n-1};q)_\infty} $$

for $m = n$ and $y = (zt^{n-1}, zt^{n-2}, \dots, zt, z)$

q,t-Littlewood–Richardson coefficients

$$P_\mu(x;q,t)P_\nu(x;q,t)=\sum_\lambda f^\lambda_{\mu\nu}(q,t)P_\lambda(x;q,t)$$

with $f^\lambda_{\mu\nu}(q,t)\in\mathbb{Q}(q,t)$

q, t-Littlewood–Richardson coefficients

$$P_\mu(x;q,t)P_\nu(x;q,t) = \sum_\lambda f^\lambda_{\mu\nu}(q,t)P_\lambda(x;q,t)$$

with $f^\lambda_{\mu\nu}(q,t) \in \mathbb{Q}(q,t)$

- (Conjecture)

  For $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ partitions with $\mu \subset \lambda$

$$\sum_\nu t^{n(\nu)} f^\lambda_{\mu\nu}(q,t) \frac{(t^{-1};q,t)_\nu}{c'_\nu(q,t)} = t^{(1-n)|\mu|+n(\lambda)} P_\mu(1,t,\cdots,t^{n-1};q,t) W^{(0)}_{\mu\lambda}(q,t) \times \frac{(qt^{n-1};q,t)_\lambda}{c'_\lambda(q,t)} \prod_{i,j=1}^n \frac{(qt^{j-i-1};q)_{\lambda_i-\mu_j}}{(qt^{j-i};q)_{\lambda_i-\mu_j}}$$

- (Conjecture')

$$\sum_{\nu} t^{n(\nu)} f^{\lambda}_{\mu\nu}(q,t) \frac{(t^{-1};q,t)_{\nu}}{c'_{\nu}(q,t)} = W^{(0)}_{\mu\lambda}(q,t) \sum_{\nu} t^{n(\nu)} f^{\lambda}_{\mu\nu}(q,t) \frac{(qt^{-1};q,t)_{\nu}}{c'_{\nu}(q,t)}$$

- (Conjecture')

$$\sum_{\nu} t^{n(\nu)} f^{\lambda}_{\mu\nu}(q,t) \frac{(t^{-1};q,t)_\nu}{c'_\nu(q,t)} = W^{(0)}_{\mu\lambda}(q,t) \sum_{\nu} t^{n(\nu)} f^{\lambda}_{\mu\nu}(q,t) \frac{(qt^{-1};q,t)_\nu}{c'_\nu(q,t)}$$

- Conjecture (or Conjecture') proves the $m=n$ case of the $\mathfrak{sl}_3$ q-binomial theorem (versions I and II).
  - True for $n=m=1$ (easy)

$$f^{(\lambda_1)}_{(\mu_1,\nu_1)}(q,t) = \delta_{\mu_1+\nu_1,\lambda_1}$$

  - True for $n=m=2$ (moderately hard)

$$f^{(\lambda_1,\lambda_2)}_{(\mu_1,\mu_2),(\nu_1,\nu_2)}(q,t) = \frac{(t, t^2 q^{\lambda_1-\lambda_2}, q^{\lambda_1-\mu_1-\nu_2+1}, q^{\mu_1-\lambda_2+\nu_2+1}; q)_{\lambda_2-\mu_2-\nu_2}}{(q, tq^{\lambda_1-\lambda_2+1}, tq^{\lambda_1-\mu_1-\nu_2}, tq^{\mu_1-\lambda_2+\nu_2}; q)_{\lambda_2-\mu_2-\nu_2}}$$

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Merci!