The University of Melbourne
Department of Mathematics and Statistics
Assessment, 2nd Semester, 2005

620-463
Network Optimisation

Student Name:
Student Number:

(1) This examination will contribute 40% to your assessment.

(2) This paper consists of 8 questions and 3 pages, including this cover page.

(3) You will have 7 days working on these questions, from Friday 4 November to Friday 11 November 2005. Please hand in your solutions together with this exam paper by Friday 11 November.

(4) Please write your answers in a clear, logical, precise and succinct way. Unclear writing may result in deduction of your marks.

(5) This must be your own work. Please do not consult, collaborate or discuss these questions with anyone.
1. Let $G = (V, E)$ be an undirected, connected graph, and let $c : E \to \mathbb{R}$ be a weight function on $E$. For any spanning tree $T$ of $G$, define

$$M(T; c) = \max_{e \in E(T)} c_e$$

to be the maximum weight of an edge of $T$ under $c$. Then define

$$m = \min_T M(T; c)$$

with $T$ running over all spanning trees of $G$. Prove that, for any minimum spanning tree $T$ of $(G, c)$, we must have $M(T; c) = m$.

2. Let $G = (V, E)$ be a directed graph and $s, t$ be distinct vertices of $G$. Suppose that to each arc $e \in E$ we assign a number $r_e$, $0 < r_e \leq 1$, to represent its reliability. The reliability of a directed path is defined to be the product of the reliabilities of the arcs on the path. The problem is to find a directed $(s, t)$-path with maximum reliability. Reduce this problem to an ordinary shortest path problem, and justify your reduction.

3. Let $(G, u, s, t)$ be a network, where as usual $G = (V, E)$ is a directed graph, $u : E \to \mathbb{R}_+$ is a capacity function, and $s, t$ are the source and sink respectively. Prove that, if an arc $e \in E$ with $u_e > 0$ is saturated in every maximum flow $x$ (that is, $x_e = u_e$), then there exists a minimum cut $\delta^+(S)$ such that $e \in \delta^+(S)$. (Hint: You may prove this by contradiction, using the max-flow min-cut theorem. Reduce the capacity of $e$ by $\varepsilon$ for some $0 < \varepsilon < u_e$ to get a new network.)

4. Prove that, in using the Network Simplex Method to solve an instance $(G, u, b, c)$ of the (general) Minimum Cost Flow Problem, a spanning tree structure $(T, L, U)$ determines uniquely its tree solution $x$. Give this tree solution explicitly in terms of $u$ and $b$.

5. Let $(G, u, b, c)$ be an instance of the minimum cost flow problem. Suppose we multiply the cost of each arc by a positive constant $k$. Denote by $c'$ the new cost function, so that $c'_e = kc_e$ for each arc $e$ of $G$. Using the optimal conditions in the
lecture notes, prove that \((G, u, b, c)\) and \((G, u, b, c')\) have the same set of optimal solutions.

6. This problem is about the Network Simplex Method for general Minimum Cost Flow Problem and we use the notation in the lecture notes. Show that, in an iteration of this algorithm with \(\varepsilon(x) = 0\), if both the old and new tree structures are strongly feasible, then \(\sum_{v \in V} y_v\) decreases strictly.

7. Let \(G = (V, E)\) be an undirected graph and \(M\) a matching of \(G\).
   
   (i) Show that, if a vertex \(v\) is \(M\)-covered and \(M'\) is obtained from \(M\) by augmenting along some \(M\)-augmenting path, then \(v\) is \(M'\)-covered.

   (ii) A subset \(S \subseteq V\) is called matchable if each vertex in \(S\) is covered by some matching of \(G\). Using (i) and the Blossom Algorithm, prove that each vertex in a matchable subset \(S\) is covered by some maximum matching of \(G\).

8. A matching \(M\) in an undirected graph \(G = (V, E)\) is maximal if for every edge \(vw \in E - M, M \cup \{vw\}\) is not a matching. Prove that \(|M| \geq \nu(G)/2\) for any maximal matching \(M\), where \(\nu(G)\) is the number of edges in a maximum matching of \(G\).