1. Rigorously prove that $3n^3 + 2n^2(\log n)^{99}$ is in $O(n^3)$.

2. Is $2^n = \Theta(3^n)$?

3. Rigorously prove that $(n+a)^b = \Theta(n^b)$ for all constants $a$ and $b > 0$.

4. Rank the following running times for an algorithm:
   \[ O(n!), O(n), (n \log n), O(2^n), O(2^{\sqrt{n}}), O(n^{100}), O(2^{n^2}) \]

5. A graph $G$ is bipartite if $V(G)$ can be partitioned into subsets $A$ and $B$ such that every edge has one endpoint in $A$ and one endpoint in $B$.

   Prove that a graph $G$ is bipartite if and only if $G$ has no odd cycle.

6. Write a polynomial algorithm to test whether a given graph is bipartite.

7. Consider the 2-Satisfiability problem:
   \[ \text{input: boolean expression } B \text{ in which each clause has exactly two literals} \]
   \[ \text{question: is } B \text{ satisfiable?} \]

   Write a polynomial algorithm for this problem. Hints appear on the reverse page.

8. Show that in the reduction from SAT to 3-SAT, the clause
   \[ C_i = \lambda_1 \lor \lambda_2 \lor \cdots \lor \lambda_\ell \]

   is satisfiable if and only if the $\ell - 2$ clauses
   \[
   (\lambda_1 \lor \lambda_2 \lor x_1) \land \\
   (\bar{x}_1 \lor \lambda_3 \lor x_2) \land (\bar{x}_2 \lor \lambda_4 \lor x_3) \land \cdots \land (\bar{x}_{\ell-4} \lor \lambda_{\ell-2} \lor x_{\ell-3}) \land \\
   (\bar{x}_{\ell-3} \lor \lambda_{\ell-1} \lor \lambda_\ell)
   \]

   are satisfiable, where $x_1, \ldots, x_{\ell-3}$ are new variables.
Hints for question 7:

(a) Model the problem using a directed graph $G$ that has two vertices $x_T$ and $x_F$ for each variable $x$: the vertex $x_T$ represents the event that $x$ is assigned TRUE, and $x_F$ represents the event that $x$ is assigned FALSE.

(b) Have a directed edge from a vertex $v$ to a vertex $w$ whenever the event corresponding to $v$ implies the event corresponding to $w$. Write a formal definition of $G$.

(c) What property of the graph characterises when $B$ is satisfiable? Can we test this property in polynomial time?