1. Let $G$ be a graph. A set $S \subseteq V(G)$ is a *stable set* if no edge of $G$ has both endpoints in $S$. Let $\mathcal{J}$ be the set of all stable sets in $G$. Then $(V(G), \mathcal{J})$ is an independence system. Why? Give an example of a graph $G$ such that $(V(G), \mathcal{J})$ is not a matroid.

Note that the problem: “given a graph $G$ and an integer $k$, does $G$ contain a stable set of size at least $k$” is NP-complete. Why is this relevant to the first question?

2. Let $G$ be a graph. Let $\mathcal{J}$ be the set of all matchings in $G$. Then $(E(G), \mathcal{J})$ is an independence system. Why? Give an example of a graph $G$ such that $(E(G), \mathcal{J})$ is not a matroid.

3. Jobs labelled 1, 2, \ldots, $n$ are to be processed by a single machine. All jobs require the same processing time. Each job $j$ has a deadline $d_j$ and a profit $c_j$, which will be earned if the job is completed by its deadline. Develop a polynomial time algorithm that will find the ordering of the jobs that maximises total profit. Prove the correctness of your algorithm.

*Hint #1:* First, prove that if $X$ is a subset of the jobs that can be completed on time, then the jobs in $X$ will be completed on time if the jobs in $X$ are processed in the order of their deadlines.

*Hint #2:* Prove that $\{(1, 2, \ldots, n), \mathcal{J}\}$ is a matroid, where

$$\mathcal{J} = \{X \subseteq \{1, 2, \ldots, n\} : \text{every job in } X \text{ can be completed on time}\}.$$

Solve the following scheduling problem. The machine is available from 12 noon. Each job takes one hour.

<table>
<thead>
<tr>
<th>job $j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>profit $c_j$</td>
<td>20</td>
<td>15</td>
<td>10</td>
<td>10</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>deadline $d_j$</td>
<td>3pm</td>
<td>1pm</td>
<td>2pm</td>
<td>1pm</td>
<td>2pm</td>
<td>5pm</td>
<td>5pm</td>
<td>4pm</td>
<td>2pm</td>
<td>6pm</td>
</tr>
</tbody>
</table>

4. (Challenging) Let $G$ be a graph, such that each edge of $G$ is assigned a colour. Characterise when $G$ has a spanning tree with all its edges coloured differently. That is, prove a theorem that says $G$ has a spanning tree with all its edges coloured differently if and only if .............

*Hint #1:* Suppose that $G$ has a spanning tree $T$ with all its edges coloured differently. Let $F$ be the union of any set of $p$ colour classes in $G$. If $G - F$ has $\ell$ connected components, then there are at least $\ell - 1$ edges in $T \cap F$ that connect the connected components of $G - F$. Why? Since all the edges in $T \cap F$ are coloured differently, $p \geq \ell - 1$. Is this necessary condition also sufficient?

*Hint #2:* Apply the Matroid Intersection Theorem, where one matroid is the cycle matroid of $G$, and the other is the partition matroid determined by the given edge colouring.