

Graph Minors and Minimum Degree

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Abstract

Let \mathcal{D}_k be the class of graphs for which every minor has minimum degree at most k . Then \mathcal{D}_k is closed under taking minors. By the Robertson-Seymour graph minor theorem, \mathcal{D}_k is characterised by a finite family of minor-minimal forbidden graphs, which we denote by $\widehat{\mathcal{D}}_k$. This paper discusses $\widehat{\mathcal{D}}_k$ and related topics. We obtain four main results:

1. We prove that every $(k+1)$ -regular graph with less than $\frac{4}{3}(k+2)$ vertices is in $\widehat{\mathcal{D}}_k$, and this bound is best possible.
2. We characterise the graphs in $\widehat{\mathcal{D}}_{k+1}$ that can be obtained from a graph in $\widehat{\mathcal{D}}_k$ by adding one new vertex.
3. For $k \leq 3$ every graph in $\widehat{\mathcal{D}}_k$ is $(k+1)$ -connected, but for large k , we exhibit graphs in $\widehat{\mathcal{D}}_k$ with connectivity 1. In fact, we construct graphs in \mathcal{D}_k with arbitrary block structure.
4. We characterise the complete multipartite graphs in $\widehat{\mathcal{D}}_k$, and prove analogous characterisations with minimum degree replaced by connectivity, treewidth, or pathwidth.

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1 Introduction

The theory of graph minors developed by Robertson and Seymour [25] is one of the most important in graph theory influencing many branches of mathematics. Let \mathcal{X} be a minor-closed class of graphs¹. A graph G is a *minimal forbidden minor* of \mathcal{X} if G is not in \mathcal{X} but every proper minor of G is in \mathcal{X} . Let $\widehat{\mathcal{X}}$ be the set of minimal forbidden minors of \mathcal{X} . By the graph minor theorem of Robertson and Seymour [25], $\widehat{\mathcal{X}}$ is a finite set. For various minor-closed classes the list of minimal forbidden minors is known. Most famously, if \mathcal{P} is the class of planar graphs, then the Kuratowski-Wagner theorem states that $\widehat{\mathcal{P}} = \{K_5, K_{3,3}\}$. However, in general, determining the minimal forbidden minors for a particular minor-closed class is a challenging problem.

Let $\delta(G)$ be the minimum degree of a graph G . Let \mathcal{D}_k be the class of graphs G such that every minor of G has minimum degree at most k . Then \mathcal{D}_k is minor-closed. Let $\widehat{\mathcal{D}}_k$ be the set of minimal forbidden minors of \mathcal{D}_k . By the graph minor theorem, $\widehat{\mathcal{D}}_k$ is finite for each k . The structure of graphs in $\widehat{\mathcal{D}}_k$ is the focus of this paper. For small values of k , it is known that $\widehat{\mathcal{D}}_0 = \{K_2\}$ and $\widehat{\mathcal{D}}_1 = \{K_3\}$ and $\widehat{\mathcal{D}}_2 = \{K_4\}$ and $\widehat{\mathcal{D}}_3 = \{K_5, K_{2,2,2}\}$; see Section 2. Determining $\widehat{\mathcal{D}}_4$ is an open problem.

The majority of this paper studies the case of general k rather than focusing on small values. Our first main result shows that, in some sense, there are many graphs in $\widehat{\mathcal{D}}_k$. In particular, every sufficiently small $(k+1)$ -regular graph is in $\widehat{\mathcal{D}}_k$. This result is proved in Section 5.

Theorem 1.1. *Every $(k+1)$ -regular graph with less than $\frac{4}{3}(k+2)$ vertices is in $\widehat{\mathcal{D}}_k$. Moreover, for all $k \equiv 1 \pmod{3}$ there is a $(k+1)$ -regular graph on $\frac{4}{3}(k+2)$ vertices that is not in $\widehat{\mathcal{D}}_k$.*

Our second main result characterises the graphs in $\widehat{\mathcal{D}}_{k+1}$ that can be obtained from a graph in $\widehat{\mathcal{D}}_k$ by adding one new vertex.

Theorem 1.2. *Let S be a set of vertices in a graph $G \in \mathcal{D}_k$. Let G' be the graph obtained from G by adding one new vertex adjacent to every vertex in S . Then $G' \in \widehat{\mathcal{D}}_{k+1}$ if and only if S is the set of vertices of degree $k+1$ in G .*

Theorem 1.2 is proved in Section 6 along with various corollaries of Theorems 1.1 and 1.2.

¹All graphs considered in this paper are undirected, simple, and finite.

To *contract* an edge vw in a graph G means to delete vw , identify v and w , and replace any parallel edges by a single edge. The contracted graph is denoted by G/vw . If $S \subseteq E(G)$ then G/S is the graph obtained from G by contracting each edge in S (while edges in S remain in G). The graph G/S is called a *contraction minor* of G .

A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. That is, H can be obtained from G by a sequence of edge contractions, edge deletions, or vertex deletions. For each vertex v of H , the set of vertices of G that are contracted into v is called a *branch set* of H . A class \mathcal{X} of graphs is *minor-closed* if every minor of every graph in \mathcal{X} is also in \mathcal{X} , and some graph is not in \mathcal{X} .

The *join* of graphs G and H , denoted by $G * H$, is the graph obtained by adding all possible edges between disjoint copies of G and H . Let \overline{G} denote the complement of a graph G .

It is natural to expect that graphs in $\widehat{\mathcal{D}}_k$ are, in some sense, highly connected. For example for $k \leq 3$ all the graphs in $\widehat{\mathcal{D}}_k$ are $(k + 1)$ -connected. However, this is not true in general. In Section 2 we exhibit a graph in $\widehat{\mathcal{D}}_4$ with connectivity 1. In fact, our third main result, proved in Section 7, constructs graphs in $\widehat{\mathcal{D}}_k$ ($k \geq 9$) with arbitrary block structure.

Theorem 1.3. *Let T be the block decomposition tree of some graph. Then for some k , T is the block decomposition tree of some graph in $\widehat{\mathcal{D}}_k$.*

A complete characterisation of graphs in $\widehat{\mathcal{D}}_k$ is probably hopeless. So it is reasonable to restrict our attention to particular subsets of $\widehat{\mathcal{D}}_k$. A graph is *complete c -partite* if the vertices can be c -coloured so that two vertices are adjacent if and only if they have distinct colours. Let K_{n_1, n_2, \dots, n_c} be the complete c -partite graph with n_i vertices in the i -th colour class. Since every graph in $\widehat{\mathcal{D}}_k$ for $k \leq 3$ is complete multipartite, it is natural to consider the complete multipartite graphs in $\widehat{\mathcal{D}}_k$. Our fourth main result characterises the complete multipartite graphs in $\widehat{\mathcal{D}}_k$.

Theorem 1.4. *For all $k \geq 1$, a complete multipartite graph G is in $\widehat{\mathcal{D}}_k$ if and only if for some $b \geq a \geq 1$ and $p \geq 2$,*

$$G = K_{\underbrace{a, b, \dots, b}_p},$$

such that $k + 1 = a + (p - 1)b$ and if $p = 2$ then $a = b$.

Theorem 1.4 is proved in Section 8. Moreover, we prove that the same characterisation holds for the minimal forbidden complete multipartite minors for the class of graphs for which every minor has connectivity at most k . And Theorem 8.9 is an analogous result for graphs of treewidth at most k and pathwidth at most k .

Finally, note that our results can be interpreted in terms of the *contraction degeneracy* of a graph G , which is defined to be the maximum, taken over all minors H of G , of the minimum degree of H . Thus, $G \in \widehat{\mathcal{D}}_k$ if and only if the contraction degeneracy of G is at most k . See [2, 3, 18, 31, 32] for results about the computational complexity of determining the contraction degeneracy, and its relation to lower bounds on treewidth.

2 Basics and Small Values of k

This section gives some basic results about $\widehat{\mathcal{D}}_k$ and reviews what is known about $\widehat{\mathcal{D}}_k$ for small values of k . We have the following characterisation of graphs in $\widehat{\mathcal{D}}_k$.

Lemma 2.1. *$G \in \widehat{\mathcal{D}}_k$ if and only if*

(D1) $\delta(G) = k + 1,$

(D2) *every proper contraction minor of G has minimum degree at most k ,*

(D3) *G is connected, and*

(D4) no two vertices both with degree at least $k + 2$ are adjacent in G .

Proof. (\implies) Suppose that $G \in \widehat{\mathcal{D}}_k$. That is, $\delta(G) \geq k + 1$ and every minor of G has minimum degree at most k . In particular, every contraction minor of G has minimum degree at most k , thus proving (D2). If G is not connected then each component of G is a proper minor with minimum degree $k + 1$. This contradiction proves (D3). If adjacent vertices v and w both have degree at least $k + 2$, then $G - vw$ is a proper minor of G with minimum degree at least $k + 1$. This contradiction proves (D4). In particular, some vertex has degree $k + 1$. Thus $\delta(G) = k + 1$ and (D1) holds.

(\impliedby) Suppose that conditions (D1)–(D4) hold. Suppose on the contrary that some proper minor of G has minimum degree at least $k + 1$. Let H be such a minor with the maximum number of edges. Since G is connected, H can be obtained by edge contractions and edge deletions only. (Deleting a non-isolated vertex v can be simulated by contracting one edge and deleting the other edges incident to v .) Condition (D4) implies that every edge has an endpoint with degree $k + 1$, implying that every proper subgraph of G has minimum degree at most k . Hence at least one edge of G was contracted in the construction of H . Since H was chosen with the maximum number of edges, no edges were deleted in the construction of H . That is, H is a contraction minor. Condition (D2) implies that H has minimum degree at most k . This contradiction proves that every proper minor of G has minimum degree at most k . Thus condition (D1) implies that $G \in \widehat{\mathcal{D}}_k$. \square

Observe that Lemma 2.1 immediately implies that for all $k \geq 0$,

$$K_{k+2} \in \widehat{\mathcal{D}}_k . \tag{1}$$

Now consider small values of k . Observe that \mathcal{D}_0 is the class of edgeless graphs, and $\widehat{\mathcal{D}}_0 = \{K_2\}$. Similarly \mathcal{D}_1 is the class of forests, and $\widehat{\mathcal{D}}_1 = \{K_3\}$. Graphs in \mathcal{D}_2 are often called *series-parallel*. $\widehat{\mathcal{D}}_2$ and $\widehat{\mathcal{D}}_3$ are easily determined; see Figure 1.

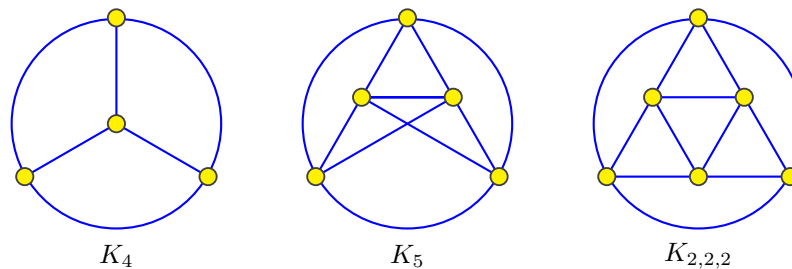


Figure 1: Graphs in $\widehat{\mathcal{D}}_2$ and $\widehat{\mathcal{D}}_3$.

Proposition 2.2. $\widehat{\mathcal{D}}_2 = \{K_4\}$.

Proof. By (1), $K_4 \in \widehat{\mathcal{D}}_2$. Consider $G \in \widehat{\mathcal{D}}_2$. By Lemma 2.1, G has minimum degree 3. Dirac [9] proved that every graph with minimum degree at least 3 contains a K_4 -minor; also see [16, 27, 33, 34]. Thus G contains a K_4 -minor. If $G \not\cong K_4$, then the K_4 -minor in G is not proper, implying $G \notin \widehat{\mathcal{D}}_2$ by Lemma 2.1. Hence $G \cong K_4$. \square

Proposition 2.3. $\widehat{\mathcal{D}}_3 = \{K_5, K_{2,2,2}\}$.

Proof. By (1), $K_5 \in \widehat{\mathcal{D}}_3$. Since $K_{2,2,2}$ is planar, every proper minor of $K_{2,2,2}$ is a planar graph on at most five vertices, which by Euler's Formula, has a vertex of degree at most 3. Thus $K_{2,2,2} \in \widehat{\mathcal{D}}_3$.

Consider $G \in \widehat{\mathcal{D}}_3$. By Lemma 2.1, G has minimum degree 4. In Appendix A we prove that every graph with minimum degree at least 4 contains a 4-connected minor². Halin and Jung [17] proved that every 4-connected graph contains K_5 or $K_{2,2,2}$ as a minor. Thus G contains K_5 or $K_{2,2,2}$ as a minor. Suppose on the contrary that G is isomorphic to neither K_5 nor $K_{2,2,2}$. Then G contains K_5 or $K_{2,2,2}$ as a proper minor. Thus G contains a proper minor with minimum degree 4, implying $G \notin \widehat{\mathcal{D}}_4$ by Lemma 2.1. Hence G is isomorphic to K_5 or $K_{2,2,2}$. \square

Determining $\widehat{\mathcal{D}}_4$ is an open problem. But we do know nine graphs in $\widehat{\mathcal{D}}_4$, as illustrated in Figure 2. One of these graphs is the *icosahedron* I , which is the unique 5-regular planar triangulation (on twelve vertices). Mader [21] proved that every planar graph with minimum degree 5 contains I as a minor. More generally, Mader [21] proved that every graph with minimum degree at least 5 contains a minor in $\{K_6, I, C_5 * \overline{K_3}, K_{2,2,2,1} - e\}$, where e is an edge incident to the degree-6 vertex in $K_{2,2,2,1}$. However, since $K_{2,2,2,1} - e$ has a degree-4 vertex, it is not in $\widehat{\mathcal{D}}_4$. Fijavž [11] proved that every graph on at most 9 vertices with minimum degree at least 5 contracts to K_6 , $K_{2,2,2,1}$ or $C_5 * \overline{K_3}$. The graphs G_1 and G_2 are discussed further in Section 3. The graphs D_1 and D_3 are due to Fijavž [11], while D_2 is due to Mader [21]. Note that D_1 , D_2 and D_3 are not 5-connected. In fact, D_3 has a cut-vertex. It is an example of a more general construction given in Section 7. In the language used there, D_3 is obtained from two copies of the single-horned graph $G_{5,4}$ by identifying the two horns.

Proposition 2.4. $\{K_6, I, C_5 * \overline{K_3}, K_{1,2,2,2}, G_1, G_2, D_1, D_2, D_3\} \subseteq \widehat{\mathcal{D}}_4$.

Proof. This result was verified by computer. (The code is available from the authors upon request.) We now give manual proofs for some of these graphs.

$K_6 \in \widehat{\mathcal{D}}_4$ by (1).

I is not in \mathcal{D}_4 since it is 5-regular. Every proper minor of I is planar with at most eleven vertices. By Euler's Formula, every such graph has minimum degree at most 4, and is thus in \mathcal{D}_4 . Hence $I \in \widehat{\mathcal{D}}_4$.

²This result was attributed by Maharry [23] to Halin and Jung [17]. While the authors acknowledge their less than perfect understanding of German, Halin and Jung actually proved that every 4-connected graph contains K_5 or $K_{2,2,2}$ as a minor. This is confirmed by Tutte's review of the Halin and Jung paper in MathSciNet.

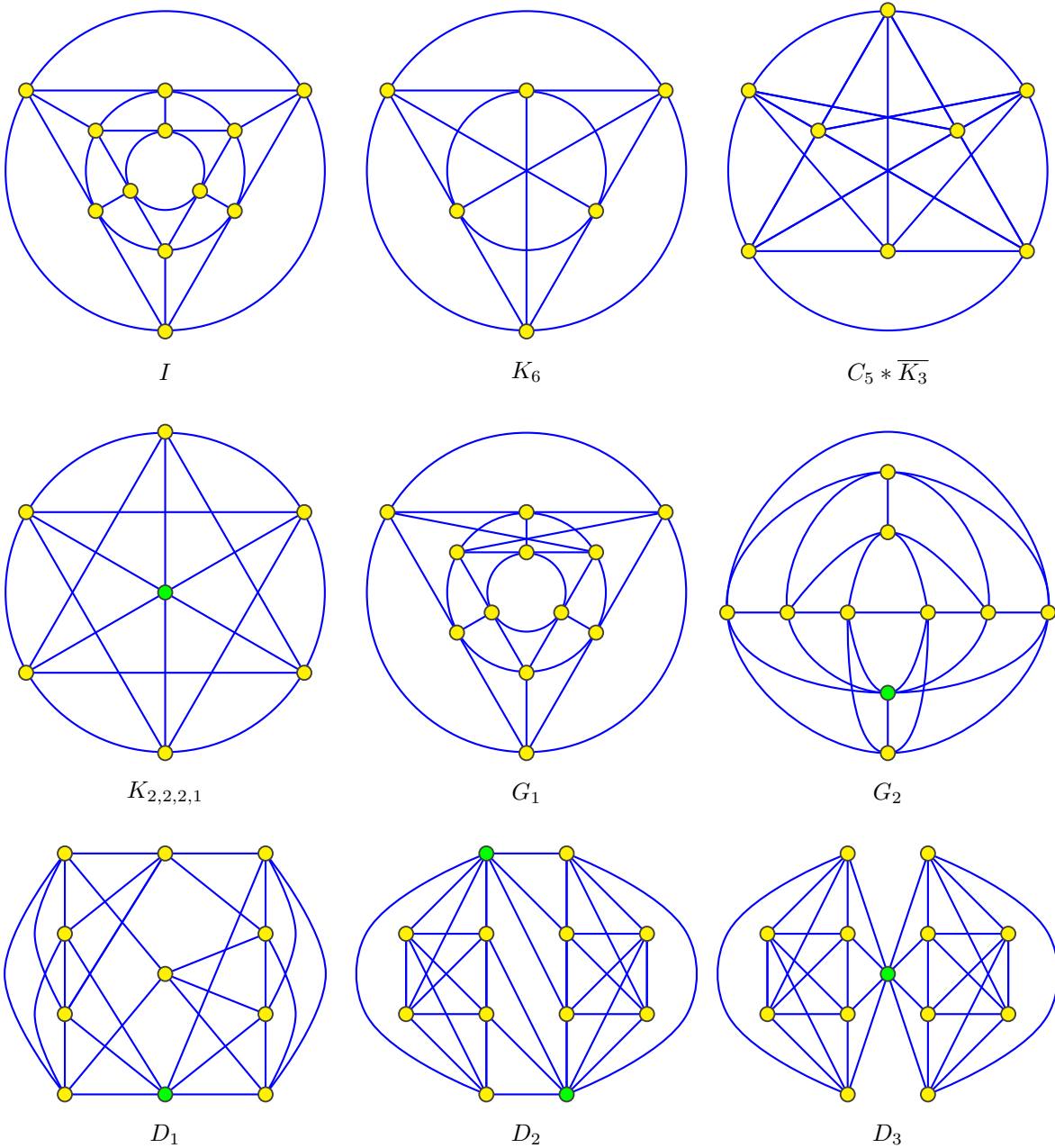


Figure 2: The known graphs in $\widehat{\mathcal{D}}_4$; vertices with degree more than 5 are highlighted.

We now prove that $C_5 * \overline{K_3} \in \mathcal{D}_4$. Since $C_5 * \overline{K_3}$ is 5-regular, conditions (D1), (D3) and (D4) hold in Lemma 2.1. Suppose that $C_5 * \overline{K_3}$ contains a proper contraction minor H with $\delta(H) \geq 5$. Thus $|V(H)| \geq 6$, and H was obtained by at most two edge contractions. Since every edge of $C_5 * \overline{K_3}$ is in a triangle with a degree-5 vertex, H was obtained by exactly two edge contractions. Since each edge in the C_5 part of $C_5 * \overline{K_3}$ is in three triangles, no edge in the C_5 part was contracted. Thus one contracted edge was vw where

$v \in C_5$ and $w \in \overline{K_3}$. Observe that vw is in two triangles $vw x$ and $vw y$, where x and y are the neighbours of v in C_5 . Since both x and y have degree 4 in G/vw , some edge incident to x and some edge incident to y is contracted in H . This is impossible since x and y are not adjacent, and only one contraction besides vw is allowed. This contradiction proves that every proper contraction minor of G has minimum degree at most 4. Thus condition (D2) holds for $C_5 * \overline{K_3}$, and $C_5 * \overline{K_3} \in \widehat{\mathcal{D}}_4$.

That $K_{1,2,2,2}$ is in $\widehat{\mathcal{D}}_4$ follows from Theorem 8.4 with $a = 1$ and $b = 2$ and $p = 3$.

We now prove that $D_3 \in \mathcal{D}_4$. Observe that conditions (D1), (D3) and (D4) in Lemma 2.1 hold for D_3 . Suppose that D_3 contains a proper contraction minor H with $\delta(H) \geq 5$. Thus $H = D_3/S$ for some $S \subseteq E(G)$ such that $|V(H)| = 13 - |S|$. Let v be the cut-vertex in D_3 . Let G_1 and G_2 be the subgraphs of D_3 such that $D_3 = G_1 \cup G_2$ where $V(G_1) \cap V(G_2) = \{v\}$. Let $S_i := S \cap E(G_i)$. We have $|S_i| \leq |V(G_i)| - 1 = 6$. Every edge of D_3 is in a triangle with a vertex distinct from v . Thus, if $|S_i| = 1$ then some vertex in H has degree at most 4, which is a contradiction. If $2 \leq |S_i| \leq 5$ then G_i/S has at least two and at most five vertices, and every vertex in G_i/S (except possibly v) has degree at most 4, which is a contradiction. Thus $|S_i| \in \{0, 6\}$. Now $|S_1| + |S_2| = |S| \leq 7$, as otherwise H has at most five vertices. Thus $|S_1| = 0$ and $|S_2| = 6$ without loss of generality. Hence $H \cong G_1$, in which v has degree 4, which is a contradiction. Thus condition (D2) holds for D_3 . Hence $D_3 \in \widehat{\mathcal{D}}_4$. \square

3 A General Setting

The following general approach for studying minor-closed class was introduced by Fijavž [11]. A *graph parameter* is a function f that assigns a non-negative integer $f(G)$ to every graph G , such that for every integer k there is some graph G for which $f(G) \geq k$. Examples of graph parameters include minimum degree δ , maximum degree Δ , (vertex-) connectivity κ , edge-connectivity λ , chromatic number χ , clique number ω , independence number α , treewidth tw , and pathwidth pw ; see [8] for definitions.

For a graph parameter f and a graph G , let $\widehat{f}(G)$ be the maximum of $f(H)$ taken over all minors H of G . Then \widehat{f} also is a graph parameter³. For example, $\widehat{\omega}(G)$ is the order of the largest clique minor in G , often called the *Hadwiger number* of G . Let

$$\mathcal{X}_{f,k} := \{G : \widehat{f}(G) \leq k\} .$$

That is, $\mathcal{X}_{f,k}$ is the class of graphs G such that $f(H) \leq k$ for every minor H of G . Then $\mathcal{X}_{f,k}$ is minor-closed, and the set $\widehat{\mathcal{X}}_{f,k}$ of minimal forbidden minors is finite.

We have the following characterisation of graphs in $\widehat{\mathcal{X}}_{f,k}$, analogous to Lemma 2.1.

Lemma 3.1. *$G \in \widehat{\mathcal{X}}_{f,k}$ if and only if $f(G) \geq k + 1$ and every proper minor H of G has $f(H) \leq k$.*

³Let $\overline{f}(G)$ be the minimum of $f(H)$ where G is a minor of H . Then the class of graphs G with $\overline{f}(G) \leq k$ is minor-closed, and we can ask the same questions for \overline{f} as for \widehat{f} . For example, the minor crossing number [4, 5, 6] fits into this framework.

Proof. By definition, $G \in \widehat{\mathcal{X}}_{f,k}$ if and only if $G \notin \mathcal{X}_{f,k}$ but every proper minor of G is in $\mathcal{X}_{f,k}$. That is, there exists a minor H of G with $f(H) \geq k + 1$, but every proper minor H of G has $f(H) \leq k$. Thus the only minor H of G with $f(H) \geq k + 1$ is G itself. \square

Lemma 3.2. *Let α and β be graph parameters such that $\alpha(G) \leq \beta(G)$ for every graph G . Then $\mathcal{X}_{\beta,k} \subseteq \mathcal{X}_{\alpha,k}$ and $\{G : G \in \widehat{\mathcal{X}}_{\beta,k}, \alpha(G) \geq k + 1\} \subseteq \widehat{\mathcal{X}}_{\alpha,k}$.*

Proof. For the first claim, let G be a graph in $\mathcal{X}_{\beta,k}$. Then $\beta(H) \leq k$ for every minor H of G . By assumption, $\alpha(H) \leq \beta(H) \leq k$. Hence $G \in \mathcal{X}_{\alpha,k}$, implying $\mathcal{X}_{\beta,k} \subseteq \mathcal{X}_{\alpha,k}$.

For the second claim, suppose that $G \in \widehat{\mathcal{X}}_{\beta,k}$ and $\alpha(G) \geq k + 1$. By Lemma 3.1 applied to β , $\beta(G) \geq k + 1$ and every proper minor H of G has $\beta(H) \leq k$. By assumption, $\alpha(H) \leq \beta(H) \leq k$. Since $\alpha(G) \geq k + 1$, Lemma 3.1 applied to α implies that $G \in \widehat{\mathcal{X}}_{\alpha,k}$. \square

Recall that δ and κ are the graph parameters minimum degree and connectivity. Observe that $\mathcal{D}_k = \mathcal{X}_{\delta,k}$. Let

$$\mathcal{C}_k := \mathcal{X}_{\kappa,k}$$

be the class of graphs for which every minor has connectivity at most k . For $k \leq 3$, we have $\mathcal{C}_k = \mathcal{D}_k$ and $\widehat{\mathcal{C}}_k = \widehat{\mathcal{D}}_k$. That is, $\widehat{\mathcal{C}}_1 = \{K_3\}$, $\widehat{\mathcal{C}}_2 = \{K_4\}$, and $\widehat{\mathcal{C}}_3 = \{K_5, K_{2,2,2}\}$. Determining $\widehat{\mathcal{C}}_4$ is an open problem; Fijavž [11] conjectured that $\widehat{\mathcal{C}}_4 = \{K_6, I, C_5 * \overline{K_3}, K_{1,2,2,2}, G_1, G_2\}$.

Dirac [10] proved that every 5-connected planar graph contains the icosahedron as a minor (which, as mentioned earlier, was generalised by Mader [21] for planar graphs of minimum degree 5). Thus the icosahedron is the only planar graph in $\widehat{\mathcal{C}}_4$. Fijavž [12] determined the projective-planar graphs in $\widehat{\mathcal{C}}_4$ to be $\{K_6, I, G_1, G_2\}$. Fijavž [14] determined the toroidal graphs in $\widehat{\mathcal{C}}_5$ to be $\{K_7, K_{2,2,2,2}, K_{3,3,3}, K_9 - C_9\}$. See [13, 15] for related results. Also relevant is the large body of literature on contractibility; see the surveys [19, 22].

Let

$$\mathcal{T}_k := \{G : \text{tw}(G) \leq k\} \quad \text{and} \quad \mathcal{P}_k := \{G : \text{pw}(G) \leq k\}$$

respectively be the classes of graphs with treewidth and pathwidth at most k . Since treewidth and pathwidth are minor-closed, $\mathcal{T}_k = \mathcal{X}_{\text{tw},k}$ and $\mathcal{P}_k = \mathcal{X}_{\text{pw},k}$. We have

$$\kappa(G) \leq \delta(G) \leq \text{tw}(G) \leq \text{pw}(G)$$

for every graph G ; see [1, 8]. Thus Lemma 3.2 implies that

$$\mathcal{P}_k \subseteq \mathcal{T}_k \subseteq \mathcal{D}_k \subseteq \mathcal{C}_k,$$

and

$$\{G : G \in \widehat{\mathcal{D}}_k, \kappa(G) \geq k + 1\} \subseteq \widehat{\mathcal{C}}_k \tag{2}$$

$$\{G : G \in \widehat{\mathcal{T}}_k, \delta(G) \geq k + 1\} \subseteq \widehat{\mathcal{D}}_k \tag{3}$$

$$\{G : G \in \widehat{\mathcal{P}}_k, \text{tw}(G) \geq k + 1\} \subseteq \widehat{\mathcal{T}}_k. \tag{4}$$

Thus the $(k + 1)$ -connected graphs that we show are in $\widehat{\mathcal{D}}_k$ are also in $\widehat{\mathcal{C}}_k$. In particular, Theorem 1.1 implies:

Theorem 3.3. *Every $(k + 1)$ -connected $(k + 1)$ -regular graph with less than $\frac{4}{3}(k + 2)$ vertices is in $\widehat{\mathcal{C}}_k$.*

The relationship between $\widehat{\mathcal{C}}_k$ and $\widehat{\mathcal{D}}_k$ is an interesting open problem.

Open Problem 3.4. Is $\widehat{\mathcal{C}}_k \subseteq \widehat{\mathcal{D}}_k$ for all k ? Is $\widehat{\mathcal{C}}_k = \{G : G \in \widehat{\mathcal{D}}_k, \kappa(G) = k + 1\}$ for all k ?

Note that $\widehat{\mathcal{D}}_4 \neq \widehat{\mathcal{C}}_4$ since there are graphs in $\widehat{\mathcal{D}}_4$ with connectivity 1; see Section 7.

4 General Values of k

Let G be a graph. A vertex of G is *low-degree* if its degree equals the minimum degree of G . A vertex of G is *high-degree* if its degree is greater than the minimum degree of G . Recall that every graph in $\widehat{\mathcal{D}}_k$ has minimum degree $k + 1$. Thus a vertex of degree $k + 1$ in a graph in $\widehat{\mathcal{D}}_k$ is low-degree; every other vertex is high-degree. Lemma 2.1 implies that for every graph $G \in \widehat{\mathcal{D}}_k$, the high-degree vertices in G form an independent set.

Proposition 4.1. *Every graph $G \in \widehat{\mathcal{D}}_k$ has at least $k + 2$ low-degree vertices (of degree $k + 1$).*

Proof. Suppose on the contrary that G has at most $k + 1$ low-degree vertices. By Lemma 2.1, each high-degree vertex is only adjacent to low-degree vertices. Since a high-degree vertex has degree at least $k + 2$, there are no high-degree vertices. Thus G has at most $k + 1$ vertices. Thus G has maximum degree at most k , which is a contradiction. \square

For a set S of vertices in a graph G , a *common neighbour* of S is a vertex in $V(G) - S$ that is adjacent to at least two vertices in S . A *common neighbour* of an edge vw is a common neighbour of $\{v, w\}$. Common neighbours are important because of the following observation.

Observation 4.2. *Let vw be an edge of a graph G with p common neighbours. Let H be the graph obtained from G by contracting vw into a new vertex x . Then*

$$\deg_H(x) = \deg_G(v) + \deg_G(w) - p - 2.$$

For every common neighbour y of vw ,

$$\deg_H(y) = \deg_G(y) - 1.$$

For every other vertex z of H ,

$$\deg_H(z) = \deg_G(z).$$

Proposition 4.3. *For every graph $G \in \widehat{\mathcal{D}}_k$, every edge vw of G has a low-degree common neighbour.*

Proof. If $k = 1$ then $G = K_3$ and the result is trivial. Now assume that $k \geq 2$.

Suppose on the contrary that for some edge vw of G , every common neighbour of vw (if any) is high-degree. By Lemma 2.1, at least one of v and w is low-degree (with degree $k + 1$). Thus v and w have at most k common neighbours. Let u_1, \dots, u_p be the common neighbours of v and w , where $0 \leq p \leq k$.

Let H be the graph obtained from G by contracting vw into a new vertex x . The degree of each vertex of G is unchanged in H , except for v , w , and each u_i . Since $\deg_G(u_i) \geq k + 2$, we have $\deg_H(u_i) \geq k + 1$. By Observation 4.2,

$$\deg_H(x) = \deg_G(v) + \deg_G(w) - p - 2 \geq 2(k + 1) - p - 2 = 2k - p .$$

Thus if $p \leq k - 1$ then $\deg_H(x) \geq k + 1$ and H is a proper minor of G with minimum degree at least $k + 1$, implying $G \notin \widehat{\mathcal{D}}_k$.

Otherwise $p = k$, implying both v and w are low-degree vertices whose only neighbours are each other and the high-degree vertices u_1, \dots, u_k . Let J be the graph obtained from G by contracting v, w, u_1 into a new vertex y . Since each neighbour of v is high-degree and each neighbour of w is high-degree, if a vertex (other than v, w, u_1) is adjacent to at least two of v, w, u_1 then it is high-degree. Since no two high-degree vertices are adjacent, the only vertices (other than v, w, u_1) that are adjacent to at least two of v, w, u_1 are u_2, \dots, u_k . Thus every vertex in J (possibly except y) has degree at least $k + 1$. Now u_1 has at least k neighbours in G outside of $\{v, w, u_2, \dots, u_k\}$. Thus $\deg_J(y) \geq k + (k - 1) \geq k + 1$, and J is a proper minor of G with minimum degree at least $k + 1$, implying $G \notin \widehat{\mathcal{D}}_k$. \square

The next result says that for graphs in $\widehat{\mathcal{D}}_k$, every sufficiently sparse connected induced subgraph has a common neighbour.

Proposition 4.4. *For every graph $G \in \widehat{\mathcal{D}}_k$, for every connected induced subgraph H of G with n vertices and $m \leq \frac{1}{2}(k + 1)(n - 1)$ edges, there exists a vertex x in $G - H$ adjacent to at least $\deg_G(x) - k + 1 \geq 2$ vertices in H .*

Proof. Suppose that for some connected induced subgraph H with n vertices and $m \leq \frac{1}{2}(k + 1)(n - 1)$ edges, every vertex x in $G - H$ is adjacent to at most $\deg_G(x) - k$ vertices in H . Let G' be the graph obtained from G by contracting H into a single vertex v . The degree of every vertex in $G - H$ is at least $\deg_G(x) - (\deg_G(x) - k) + 1 = k + 1$ in G' . Since G has minimum degree $k + 1$, we have

$$\deg_{G'}(v) = \left(\sum_{w \in V(H)} \deg_G(w) \right) - 2m \geq n(k + 1) - 2m \geq k + 1.$$

Thus G' is a proper minor of G with minimum degree at least $k + 1$. Hence $G \notin \widehat{\mathcal{D}}_k$. This contradiction proves the result. \square

Corollary 4.5. *For every graph $G \in \widehat{\mathcal{D}}_k$, for every clique C of G with at most $k + 1$ vertices, there exists a vertex in $V(G) - C$ adjacent to at least two vertices of C .*

5 Small Regular Graphs are in $\widehat{\mathcal{D}}_k$

In this section we show that every $(k+1)$ -regular graph with sufficiently few vertices is in $\widehat{\mathcal{D}}_k$. Moreover, the bound on the number of vertices is tight.

Lemma 5.1. *Let G be a connected $(k+1)$ -regular graph on n vertices. If every edge of G is in at least $2n - 2k - 5$ triangles, then $G \in \widehat{\mathcal{D}}_k$.*

Proof. By assumption, conditions (D1), (D3) and (D4) of Lemma 2.1 are satisfied by G . Suppose on the contrary that H is a contraction minor of G with minimum degree at least $k+1$. Let S be the set of vertices of G that are incident to an edge contracted in the construction of H . Let vw be one such edge. We have $|S| \leq 2(n - |V(H)|) \leq 2n - 2k - 4$. By assumption, there is a set T of vertices of G that are adjacent to both v and w , and $|T| \geq 2n - 2k - 5 \geq |S| - 1$. Thus there is at least one vertex $x \in T - (S - \{v, w\})$, which is a vertex of H . Since x is adjacent to both endpoints of the contracted edge vw , $\deg_H(x) \leq k$. This contradiction proves condition (D2) for G . Lemma 2.1 implies that $G \in \widehat{\mathcal{D}}_k$. \square

Lemma 5.2. *For every $(k+1)$ -regular graph G on n vertices, every edge vw of G is in at least $2k + 2 - n$ triangles.*

Proof. Say vw is in t triangles. Thus v and w have t common neighbours. Thus v has $k - t$ neighbours not adjacent to w , and w has $k - t$ neighbours not adjacent to v . Thus $n \geq 2 + t + 2(k - t) = 2k + 2 - t$, implying $t \geq 2k + 2 - n$. \square

Theorem 5.3. *Every $(k+1)$ -regular graph G with $n < \frac{4}{3}(k+2)$ vertices is in $\widehat{\mathcal{D}}_k$.*

Proof. Every disconnected $(k+1)$ -regular graph has at least $2k + 4$ vertices. Since $n < 2k + 4$ we can assume that G is connected. By Lemma 5.2, every edge of G is in at least $2k + 2 - n$ triangles. Now $2k + 2 - n \geq 2n - 2k - 5$ since $n \leq \frac{1}{3}(4k + 7)$. Thus every edge of G is in at least $2n - 2k - 5$ triangles. By Lemma 5.1, $G \in \widehat{\mathcal{D}}_k$. \square

Theorem 5.3 is best possible in the following sense.

Proposition 5.4. *For all $k \equiv 1 \pmod{3}$ there is a $(k+1)$ -regular graph G on $n = \frac{4}{3}(k+2)$ vertices that is not in $\widehat{\mathcal{D}}_k$.*

Proof. Let $p := \frac{1}{3}(k+2)$. Then $p \in \mathbb{Z}$. Let G be the graph whose complement \overline{G} is the disjoint union of $K_{p,p}$ and $K_{p,p}$. Then G has $4p = n$ vertices, and every vertex has degree $n - 1 - p = k + 1$. Observe that G contains a matching M of p edges (between the two $K_{p,p}$ subgraphs in \overline{G}), such that every vertex is adjacent to at least one endpoint of every edge in M . Contracting each edge in M we obtain a K_{3p} -minor in G , which has minimum degree $k + 1$. Thus $G \notin \widehat{\mathcal{D}}_k$. \square

Theorem 5.3 can be rewritten in terms of complements.

Corollary 5.5. *If G is an r -regular graph on $n \geq 4r + 1$ vertices, then $\overline{G} \in \widehat{\mathcal{D}}_{n-r-2}$.*

6 A Construction

We now describe how a graph in $\widehat{\mathcal{D}}_{k+1}$ can be constructed from a graph in $\widehat{\mathcal{D}}_k$. Let G^+ be the graph obtained from a graph G by adding one new vertex that is adjacent to each vertex of minimum degree in G . If $G \in \widehat{\mathcal{D}}_k$ then the vertices of minimum degree are the low-degree vertices.

Lemma 6.1. *If $G \in \widehat{\mathcal{D}}_k$ then $G^+ \in \widehat{\mathcal{D}}_{k+1}$.*

Proof. Let v be the vertex of $G^+ - G$. Every low-degree vertex in G has degree $k + 1$, and thus has degree $k + 2$ in G^+ . Every high-degree vertex in G has degree at least $k + 2$, which is unchanged in G^+ . By Proposition 4.1, G has at least $k + 2$ low-degree vertices. Thus v has degree at least $k + 2$ in G^+ . Thus G^+ has minimum degree $k + 2$. Suppose on the contrary that $G \in \widehat{\mathcal{D}}_k$ but $G^+ \notin \widehat{\mathcal{D}}_{k+1}$. Thus there is a proper minor H of G^+ with minimum degree at least $k + 2$. If v is not in a branch set of H , then H is a minor of G , implying H has minimum degree at most $k + 1$, which is a contradiction. Now assume that v is in some branch set B of H . (Think of B simultaneously as a vertex of H and as a set of vertices of G^+ .) Now $H - B$ is a minor of G . If $H - B$ is G , then $B = \{v\}$ and H is not a proper minor of G^+ . Thus $H - B$ is a proper minor of G . Since $G \in \widehat{\mathcal{D}}_k$, $H - B$ has a vertex X of degree at most k . Thus X has degree at most $k + 1$ in H , which is a contradiction. \square

We also have a converse result.

Lemma 6.2. *Let S be a set of vertices in a graph $G \in \mathcal{D}_k$. Let G' be the graph obtained from G by adding one new vertex v adjacent to every vertex in S . If $G' \in \widehat{\mathcal{D}}_{k+1}$ then S is the set of low-degree vertices in G .*

Proof. Suppose that $G' \in \widehat{\mathcal{D}}_{k+1}$. If some low-degree vertex x of G is not in S , then $\deg_{G'}(x) = k + 1$ and $G' \notin \widehat{\mathcal{D}}_{k+1}$. Now assume that every low-degree vertex of G is in S . Suppose on the contrary that some high-degree vertex y of G is in S . Thus $\deg_G(y) \geq k + 2$, implying $\deg_{G'}(y) \geq k + 3$. By Proposition 4.1 there are at least $k + 2$ low-degree vertices of G , all of which are adjacent to v in G' . Thus $\deg_{G'}(v) \geq k + 3$. Hence v and y are adjacent vertices of degree at least $k + 3$ in G' . Therefore $G' \notin \widehat{\mathcal{D}}_{k+1}$ by Lemma 2.1. This contradiction proves that no high-degree vertex of G is in S . Therefore S is the set of low-degree vertices. \square

Observe that Lemmas 6.1 and 6.2 together prove Theorem 1.2. Lemma 6.1 generalises as follows. For a non-negative integer p , let G^{+p} be the graph obtained from a graph G by adding p independent vertices, each adjacent to every vertex in G .

Lemma 6.3. *Let G be a $(k + 1)$ -regular n -vertex graph in $\widehat{\mathcal{D}}_k$. Then $G^{+p} \in \widehat{\mathcal{D}}_{k+p}$ whenever $0 \leq p \leq n - k - 1$.*

Proof. Every vertex of G has degree $k + 1 + i$ in G^{+i} . Every vertex of $G^{+i} - G$ has degree n in G^{+i} . Thus, if $n > k + 1 + i$ then the vertices of minimum degree in G^{+i} are exactly the vertices of G . Thus $G^{+i} = (G^{+(i-1)})^+$ whenever $1 \leq i \leq n - k - 1$. By induction on i , applying Lemma 6.1 at each step, we conclude that $G^{+i} \in \widehat{\mathcal{D}}_{k+i}$ and $G^{+p} \in \widehat{\mathcal{D}}_{k+p}$. \square

Theorem 5.3 and Lemma 6.3 imply:

Corollary 6.4. *Let G be a $(k + 1)$ -regular graph with $n < \frac{4}{3}(k + 2)$ vertices. Then $G^{+p} \in \widehat{\mathcal{D}}_{k+p}$ whenever $0 \leq p \leq n - k - 1$.*

Corollary 6.4 implies:

Lemma 6.5. *Let $L(G)$ denote the set of minimum degree vertices in a graph G . Let $p := |G - L(G)|$. Suppose that*

- *the minimum degree of G is $k + 1$, and*
- *$|L(G)| < \frac{4}{3}(k + 2 - p)$, and*
- *$V(G) - L(G)$ is an independent set of G , and*
- *every vertex in $V(G) - L(G)$ dominates $L(G)$.*

Then $G \in \widehat{\mathcal{D}}_k$.

Proof. Let X be the subgraph of G induced by the vertices of degree $k + 1$. Thus X is $(r + 1)$ -regular, where $r = k - p$. Say X has n vertices. By assumption, $n < \frac{4}{3}(k + 2 - p) = \frac{4}{3}(r + 2)$. The high-degree vertices of G have degree n , and the low-degree vertices of G have degree $r + 1 + p$. Thus $n > r + 1 + p$. That is, $p < n - r - 1$. Thus, by Corollary 6.4, we have $G = X^{+p} \in \widehat{\mathcal{D}}_{r+p} = \widehat{\mathcal{D}}_k$. \square

7 Block Structure

In this section we show that graphs in $\widehat{\mathcal{D}}_k$ can have an arbitrary block decomposition tree⁴. Theorems 7.7 and 7.8 are the main results. Note that every graph in $\widehat{\mathcal{D}}_k$ has no cut-edge (except K_2), since a cut-edge can be contracted without decreasing the minimum degree.

A *low-high tree* is a tree T that admits a bipartition $V(T) = V_\ell \cup V_h$, such that every vertex in V_ℓ has degree at most 2, and every vertex in V_h has degree at least 2. Vertices in V_ℓ are called *low*, and vertices in V_h are called *high*. Since every leaf in a low-high tree is low, every low-high tree is a block decomposition tree.

In the following discussion, let T be a low-high tree. Let L be the set of leaves in T . Let r be an arbitrary high vertex of T , called the *root*. For each edge $vw \in E(T)$, let $\text{dist}(r, vw) := \min\{\text{dist}(r, v), \text{dist}(r, w)\}$. Let B be the set of edges of T at even distance

⁴Let G be a connected graph. Let B denote the set of blocks of G (that is, cut-edges and maximal 2-connected components). Let C denote the set of cut-vertices of G . The *block decomposition tree of G* is the tree T where $V(T) = B \cup C$, and $bc \in E(T)$ whenever the block b contains c . A *block decomposition tree* is a tree that is isomorphic to a block decomposition tree of some graph. The *bipartition* of a tree T is the partition of $V(T)$ obtained from a proper 2-colouring of T . Since every cut-vertex is contained in at least two blocks, every leaf of a block decomposition tree T belongs to the same bipartition class of T . Conversely, if a tree T admits a bipartition of its vertices such that all leaves lie in the same bipartition class, then T is a block decomposition tree.

from r . Call these edges *blue*. Similarly let $R := E(T) - B$ be the set of *red* edges in T . Since r is high and each leaf is low, each leaf is at odd distance from r . Thus each edge incident with a leaf is blue.

Lemma 7.1. *The number of blue edges $|B|$ and the number of red edges $|R|$ do not depend on the choice of r .*

Proof. It is enough to show that $|B|$ and $|R|$ do not change if we choose an alternative root r' at distance 2 from r . Let R' and B' be the sets of red and blue edges with respect to r' . Let x be the common neighbour of r and r' . Thus $rx \in B - B'$ and $r'x \in B' - B$. Apart from these edges, B and B' do not differ. Hence $|B| = |B'|$, and also $|R| = |R'|$. \square

Define

$$d := 4|L| + 2|R| .$$

Since T has at least two leaves, $d \geq 8$. By Lemma 7.1, d does not depend on the choice of r .

For each edge $e = vw$ of T such that $\text{dist}(r, v) = \text{dist}(r, w) - 1$, let T_e be the maximal rooted subtree of T containing vw , and no other neighbour of v .

Define the function $\varphi : E(T) \rightarrow \mathbb{N}$ as follows. For each blue edge e in T , define

$$\varphi(e) := 4|L \cap E(T_e)| + 2|R \cap E(T_e)| . \tag{5}$$

Now consider a red edge vw in T with $\text{dist}(r, v) = \text{dist}(r, w) - 1$. Thus $\text{dist}(r, v)$ is odd, v is low, and $\text{deg}(v) = 2$. Let uv be the blue edge incident to v . Define

$$\varphi(vw) := d + 2 - \varphi(uv) . \tag{6}$$

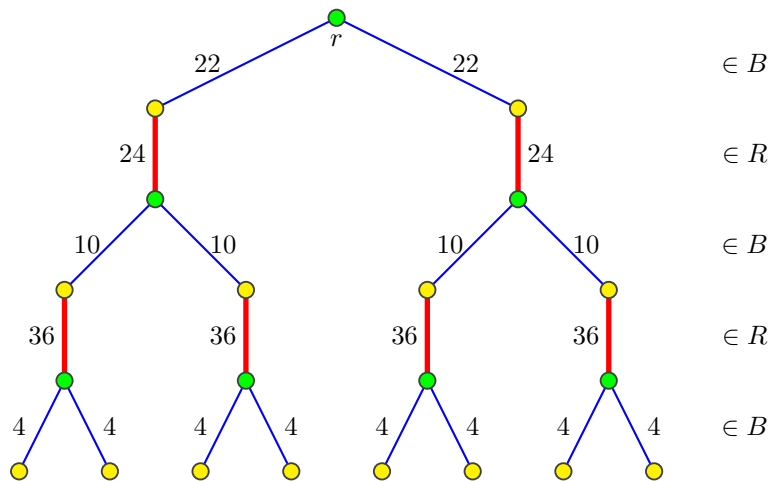


Figure 3: An example of the edge labelling φ with $|R| = 6$ and $|B| = 14$ and $|L| = 8$ and $d = 2 \cdot 6 + 4 \cdot 8 = 44$. Red edges are drawn thick.

The next lemma immediately follows from (6).

Lemma 7.2. *If v is a low vertex of degree 2 and v is incident with edges e and f , then $\varphi(e) + \varphi(f) = d + 2$.*

The sum of φ values around a high vertex is also constant.

Lemma 7.3. *Let v be a high vertex and let E_v be the set of edges incident with v . Then $\sum_{e \in E_v} \varphi(e) = d$.*

Proof. First suppose that $v = r$. Then

$$\sum_{rx \in E_r} \varphi(rx) = \sum_{rx \in E_r} 4|L \cap E(T_{rx})| + 2|R \cap E(T_{rx})| = 4|L \cap E(T)| + 2|R \cap E(T)| = d .$$

Now assume that $v \neq r$. Since v is high, $\text{dist}(v, r)$ is even, and v is incident to one red edge uv (where u is the neighbour of v closer to r than v). Thus u is low, and $\text{deg}(u) = 2$. Let t be the other neighbour of u . Let e_1, \dots, e_k be the blue edges incident to v . Then

$$\begin{aligned} \sum_{e \in E_v} \varphi(e) &= \varphi(uv) + \sum_{i=1}^k \varphi(e_i) \\ &= d + 2 - \varphi(tu) + \sum_{i=1}^k \varphi(e_i) \\ &= d + 2 - 4|L \cap E(T_{tu})| - 2|R \cap E(T_{tu})| + \sum_{i=1}^k 4|L \cap E(T_{e_i})| + 2|R \cap E(T_{e_i})| . \end{aligned}$$

Observe that $L \cap E(T_{tu}) = \bigcup_i L \cap E(T_{e_i})$ and $R \cap E(T_{tu}) - \bigcup_i R \cap E(T_{e_i}) = \{uv\}$. Thus

$$\sum_{e \in E_v} \varphi(e) = d + 2 - 2 = d .$$

□

Observe that, in principle, the definition of φ depends on the choice of r . However, this is not the case.

Lemma 7.4. *Let r and r' be high vertices of T , and let φ and φ' be the functions defined above using r and r' as roots, respectively. Then $\varphi = \varphi'$.*

Proof. Since T is connected, it is enough to show that $\varphi = \varphi'$ whenever $\text{dist}(r, r') = 2$. Let x be the common neighbour of r and r' . Let B' be the set of blue edges with respect to r' . Now B and B' (as well as R and R') differ only in rx and $r'x$. Since (5) only considers φ and φ' values of blue edges away from the root, $\varphi(e) = \varphi'(e)$ for each $e \in B \cap B'$. Since each edge incident with r or r' apart from rx and $r'x$ is in $B \cap B'$, and since d is invariant, (6) shows that φ and φ' match on every edge in $R \cap R'$. Finally Lemma 7.3 implies that φ and φ' also match on edges between rx and $r'x$. □

Lemma 7.5. $\varphi(e) \geq 4$ for every edge $e \in E(T)$.

Proof. While the colour of an edge e may depend on the choice of r , Lemma 7.4 says that $\varphi(e)$ does not depend on the choice of r . Every edge can be made blue for an appropriate choice of r , and $\varphi(e) \geq 4$ for every blue edge e by (5). \square

And now for something completely different. Let $e = u_1u_2$ and $f = u_3u_4$ be two independent edges in the complete graph K_{d+1} , where $d \geq 4$. The *single-horned graph* $G_{d,4}$ is obtained from K_{d+1} by adding a new vertex x , connecting x to u_1, u_2, u_3 , and u_4 and removing edges e and f . Observe that $\deg(x) = 4$. Call x the *horn* of $G_{d,4}$. Call the remaining vertices the *original vertices* of $G_{d,4}$, which all have degree d .

Let $a, b \geq 4$ be even integers such that $d = a + b - 2$. Choose matchings M_a and M_b with $\frac{a}{2}$ and $\frac{b}{2}$ edges, respectively, that cover all the vertices of K_{d+1} . Hence M_a and M_b share exactly one vertex. Take two new vertices x_a and x_b and join x_a to every vertex of M_a and x_b to every vertex of M_b . Next delete the edges of M_a and M_b . The resulting graph is called the *double-horned graph* $G_{d,a,b}$. As above, x_a and x_b are called the *horns* of $G_{d,a,b}$, and the remaining vertices, all of degree d , are the *original vertices*.

Let $e = uv$ be an edge in a single- or double-horned graph G . If u or v is a horn in G , then the vertex uv is a *horn* in G/e and is *original* otherwise. Inductively, we can define horns and original vertices for every contraction minor of a horned graph.

Lemma 7.6. *Let G' be a proper contraction minor of a horned graph $G_{d,4}$ or $G_{d,a,b}$. If G' contains an original vertex, then some original vertex of G' has degree less than d .*

Proof. We shall leave the proof for $G_{d,4}$ to the keen reader. Let G be the doubly horned graph $G_{d,a,b}$, and let $F \subseteq E(G)$ such that $G/F = G'$. If $|F| \geq 3$, then G' has at most d vertices, and *all* its vertices have degree less than d . Now assume that $|F| \leq 2$.

Let $e = uv$ be an edge connecting a pair of original vertices. There are at least $7 \leq a + b - 1$ original vertices in G and at least three original vertices are connected with both u and v . Thus G/e has at least three original vertices of degree less than d , which cannot all be eliminated by a single additional contraction. Hence every edge in F is incident with a horn. Let x be a horn incident with e . At least two neighbours of x (which are original vertices) have degree less than d in G/e , yet by the above argument, the edge between them cannot be contracted. \square

We are now ready to state the first theorem of this section.

Theorem 7.7. *For every low-high tree T , there is an integer d and a graph G such that:*

(G1) G is d -regular,

(G2) T is the block decomposition tree of G ,

(G3) $8 \leq d \leq 4|E(T)|$, and

(G4) $G \in \widehat{\mathcal{D}}_{d+1}$.

Proof. Adopt the above notation. Let $d := 4|L| + 2|R|$. By construction, $d \geq 8$. Note that $d \leq 4|E(T)|$ with equality only if T is a star.

For every leaf u of T , let G_u be a copy of the single-horned graph $G_{d,4}$. For every non-leaf low vertex v of T incident with edges e and f , let G_v be a copy of the double-horned graph $G_{d,a,b}$, where $a := \varphi(e)$ and $b := \varphi(f)$. Note that $a, b \geq 4$ by Lemma 7.5.

Observe that there is a natural correspondence between the set of horns in the above graphs and their degrees, and between $E(T)$ and their φ values. As illustrated in Figure 4, identifying horns wherever the edges in T have a common (high) end-vertex gives rise to a d -regular graph G (by Lemma 7.3). Hence G satisfies (G1), (G2) and (G3).

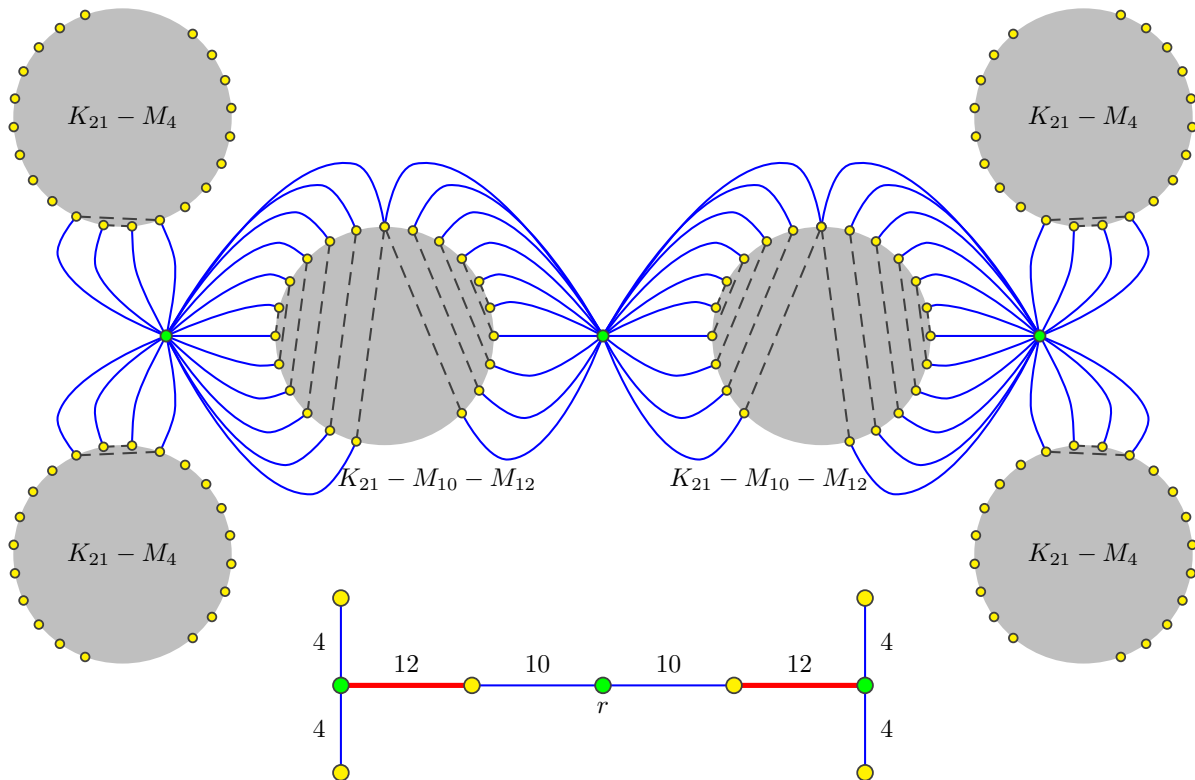


Figure 4: The graph G produced from the given low-high tree with $d = 4 \cdot 4 + 2 \cdot 2 = 20$. Shaded regions represent cliques minus the dashed matchings.

Since G is connected and d -regular, Lemma 2.1 implies that to establish (G4) it suffices to show that every proper contraction minor of G has a vertex of degree less than d . Suppose on the contrary that there is a proper contraction minor $G' = G/E'$ of G with $\delta(G) \geq d$. Take such a G' with the minimum number of vertices. Thus G' has no cut-edges, since contracting a cut-edge does not decrease δ (since $G' \not\cong K_2$).

Let H be an arbitrary block of G and consider H/E' . Suppose that H/E' is not contracted to a single vertex. Now $H/E' \not\cong K_2$ (as this would either be a nonexistent cut-edge in G' or would imply that G' has a vertex of degree 1 which is also absurd). But

if H/E' has at least three vertices and H/E' is a proper minor of H , then by Lemma 7.6, H/E' has an inner vertex of degree less than d . Hence H/E' is either trivial or is left intact in a contraction.

So we may assume that G' is obtained by shrinking several blocks of G to single vertices. We may assume that G' is obtained by first contracting $k_i \geq 0$ inner blocks of G , and later contracting $k_e \geq 0$ end-blocks of G , where $k_i + k_e \geq 1$. Let G^* be the graph obtained after contracting the inner blocks.

Now $k_i > 0$, as otherwise G' is a proper subgraph of G . By shrinking k_i inner blocks we have reduced the number of cut-vertices by k_i , and also reduced the sum of their degrees by $k_i(d + 2)$; see Lemmas 7.2 and 7.3. Hence G^* has at least one *cut-vertex* v of degree less than d , and since $G' \neq G^*$, at least one contraction of an end-block follows. Finally, contracting an end-block cannot increase $\deg(v)$. This contradiction completes the proof of (G4). \square

We now prove that minor-minimal minimum-degree graphs can have arbitrary block structure.

Theorem 7.8. *For every block decomposition tree T , there is an integer d and a graph G such that*

(H1) T is the block decomposition tree of G ,

(H2) $\delta(G) \leq 8|E(T)|$, and

(H3) $G \in \widehat{\mathcal{D}}_{d+1}$ where $d = \delta(G)$.

Proof. Let $V_c \cup V_b$ be the bipartition of $V(T)$, such that every leaf of T is in V_b . Let H_b denote the set of vertices in V_b with degree at least 3 in T . Thus T is low-high if and only if $H_b = \emptyset$. By Theorem 7.7 we may assume that T is not low-high, and $H_b \neq \emptyset$. Choose an arbitrary vertex $x \in H_b$.

Let T' be the tree obtained from T by subdividing each edge that is incident with a vertex in H_b once. Each such subdivision vertex and each vertex in $V_b - H_b$ has degree at most 2 in T' . Each vertex in $V_c \cup H_b$ has degree at least 2 in T' . Thus T' is low-high. In particular, x is a high vertex of T' .

Now $|E(T')| < 2|E(T)|$ since at least one edge of T is incident with a leaf and did not get subdivided in the construction of T' . By Theorem 7.7 there exists an integer $d' \leq 4E(T') < 8E(T)$ and a d' -regular graph $G' \in \widehat{\mathcal{D}}_{d'+1}$ such that T' is the block decomposition tree of G' . In order to keep the arguments below as simple as possible, assume that G' is the graph obtained by the construction in the proof of Theorem 7.7. Observe that every block of G' contains at least 12 vertices, since T has at least one vertex in H_b . Note that the cut-vertices of G' come in two flavours: ones that correspond to vertices of V_c , and ones that correspond to vertices of H_b . Similarly, every non-cut-vertex of G' corresponds to a vertex of $V_b - H_b$.

Now define a partition of $V(G')$ into bags $\{B_y : y \in V_b\}$ labelled by vertices V_b , satisfying the following conditions:

- (C1) for every $y \in H_b$ the bag B_y contains the cut-vertex c that corresponds to y as well as the interior vertices of every block that contains c ,
- (C2) for every $y \in V_b - H_b$ the bag B_y contains every interior vertex of a block that corresponds to $y \in H_b$.

We have so far partitioned every vertex of G' that is not a cut-vertex corresponding to a vertex in V_c .

- (C3) if c is a cut-vertex corresponding to a vertex of V_c , then let c_x be its neighbour on some shortest $c-x$ path in G' , and put c in the bag B that already contains c_x .

Observe that every block of G' contains $d' + 1$ interior vertices, hence every bag B_y contains at least $d' + 1$ vertices.

Finally we obtain G from G' by adding for each bag B_y of G' a new vertex \tilde{y} which is made adjacent to every vertex of its bag B_y . Now G' is a subgraph of G and every $v \in V(G')$ has degree equal to $d' + 1$, and new vertices have degree at least $d' + 1$. Call this process *bag extension* and let $d := d' + 1$.

Now G contains two types of blocks: *small blocks* that contain interior vertices of exactly one block of G' , and *big blocks* that contain interior vertices of several blocks of G' . Observe that every big block B contains a separating set of size two comprised of its new vertex and a vertex from H_b .

Let B' (respectively, B) be an end-block of G' (G) and let c be a cut-vertex that separates B' (B) from the rest of G' (G). By the construction of G' there are exactly four edges incident with c whose other end-vertex is in B' (B).

Let e be an arbitrary edge of G that is not one of the four edges incident to some cut-vertex of an end-block. Assume that e belongs to block B of G . Then there are at least six vertices of degree d in G that are all adjacent to both end-vertices of e . This implies that G/e contains at least six vertices of degree less than d , and no contraction of an additional two edges of B can eliminate all the vertices of degree less than d .

First observe that an end-block of G contains exactly $d + 2$ vertices and the other small blocks contain exactly $d + 3$ vertices. Every big block on the other hand contains a pair of vertices: the new vertex and a cut-vertex of G' corresponding to a vertex in H_b .

It remains to prove that $G \in \widehat{\mathcal{D}}_{d+1}$. Since every edge has an end-vertex of degree d , no edge-deleted subgraph of G has minimum degree at least d .

Hence we only have to consider contraction minors of G . Let $F \subseteq E(G)$ be a nonempty edge set and let $G^* = G/F$. We may split $F = F' \cup F^*$ so that $F' \subseteq E(G')$.

A block B of G may either get contracted to a single vertex, get partially contracted, or survive the contraction of F without changes.

First assume that B/F gets partially contracted. If B' is an end-block, then B/F has exactly $d + 1$ vertices obtained by contracting a single edge. This is not possible as a vertex of degree less than d would be created. If B is a small block, then contracting any edge of B leaves at least six vertices of degree less than d in B . Since B has $d + 3$ vertices in the beginning, an additional two contractions decrease the vertex count below $d + 1$, which is absurd.

Let B be a big block that gets partially contracted. If contraction identifies the new vertex n of B and a cut-vertex c of G' corresponding to a vertex in H_b then B/nc contains at least six vertices of degree less than d in *every* block B' of G' that is a subgraph of B . Since B' contains $d + 2$ vertices, B'/F is trivial for every $B' \subseteq B$, which is nonsense. Otherwise assume that $B' \subseteq B$ is a block of G' that gets partially contracted. The $d + 1$ interior vertices of B' are separated from the rest of G by three vertices. This implies that at most three edges are contracted in order to contract B' partially. Yet a single contraction produces six vertices of degree less than d in B' , so that an additional two contractions do not suffice.

Hence no block of G gets partially contracted in G/F . Now G/F may be obtained from G'/F by extension of bags, where G'/F is a contraction of G' that either identifies a block of G' or leaves it unchanged. In this case, G'/F contains a vertex of degree less than d' , and bag extension can only increase its degree by one. This completes the proof of Theorem 7.8. \square

Open Problem 7.9. By the Robertson-Seymour graph minor theorem, $\widehat{\mathcal{D}}_k$ is finite for each fixed k . Let $f(k)$ be the maximum number of vertices in a graph in $\widehat{\mathcal{D}}_k$. No reasonable bounds on $f(k)$ are known.

- It would be interesting to obtain a simple proof of the finiteness of $\widehat{\mathcal{D}}_k$, and to obtain good upper bounds on $f(k)$.
- By Theorem 7.7 with $T = K_{1,s}$, there is a graph $G \in \widehat{\mathcal{D}}_{4s+1}$ with $1 + s(4s + 1)$ vertices. Does every graph in $\widehat{\mathcal{D}}_k$ have $O(k^2)$ vertices?
- By Theorem 7.7 with $T = P_{2s+1}$, there is a graph $G \in \widehat{\mathcal{D}}_{2s+7}$ with diameter $2s$. Does every graph in $\widehat{\mathcal{D}}_k$ have $O(k)$ diameter?

8 Complete Multipartite Graphs

This section characterises the complete multipartite graphs in $\widehat{\mathcal{D}}_k$, in $\widehat{\mathcal{C}}_k$, in $\widehat{\mathcal{T}}_k$, and in $\widehat{\mathcal{P}}_k$. See [7, 20, 24] for other results on treewidth obstructions. We first prove three lemmas about complete multipartite graphs. The first says that complete multipartite graphs are highly connected.

Lemma 8.1. *Every complete multipartite graph G with minimum degree k is k -connected. Moreover, if vw is an edge of G such that both v and w have degree at least $k + 1$, then $G - vw$ is k -connected.*

Proof. Let x and y be distinct vertices in G . It suffices to prove that there is a set of k internally disjoint paths between x and y that avoid vw . Let R be the set of vertices coloured differently from both x and y .

First suppose that x and y have the same colour. Then $\deg(x) = \deg(y) \geq k$, and $P := \{xry : r \in R\}$ is a set of $\deg(x)$ internally disjoint paths between x and y . If vw is

in some path in P , then without loss of generality $v = x$, implying $\deg(x) \geq k + 1$, and at least k paths in P avoid vw .

Now assume that x and y are coloured differently. Let $S := \{x_1, x_2, \dots, x_p\}$ be the colour class that contains x , where $x = x_p$. Let $T := \{y_1, y_2, \dots, y_q\}$ be the colour class that contains y , where $y = y_q$. Without loss of generality, $n - p = \deg(x) \leq \deg(y) = n - q$, implying $q \leq p$. Thus

$$P := \{xy\} \cup \{xry : r \in R\} \cup \{xy_i x_i y : i \in [q - 1]\}$$

is a set of $\deg(x)$ internally disjoint paths between x and y . If $\deg(x) \geq k + 1$ then at least k paths in P avoid vw . Now assume that vw is in some path in P , but $\deg(x) = k$. Since each vertex x_i has the same degree as x , and v and w both have degree at least $k + 1$, the only possibility is that $v = y$ and $w = r$ for some $r \in R$ (or symmetrically $w = y$ and $v = r$). Thus $\deg(x) < \deg(y)$ and $q < p$. Replace the path xry in P by the path $rxr_{p-1}y$, which is internally disjoint from the other paths in P . \square

Lemma 8.2. *Let G be a complete multipartite graph on n vertices. Then*

$$\kappa(G) = \delta(G) = \text{tw}(G) = \text{pw}(G) = n - \alpha(G).$$

Proof. The degree of a vertex v equals n minus the size of the colour class that contains v . Since every independent set is contained within a colour class, the size of the largest colour class equals $\alpha(G)$. Thus $\delta(G) = n - \alpha(G)$. We have $\kappa(G) \leq \delta(G) \leq \text{tw}(G) \leq \text{pw}(G)$ for every graph G ; see [1, 8]. By Lemma 8.1, $\kappa(G) \geq \delta(G)$. Thus it suffices to prove that $\delta(G) \geq \text{pw}(G)$ for every complete multipartite graph G . Let $S = \{v_1, \dots, v_{\alpha(G)}\}$ be a largest colour class in G . Let $X := V(G) - S$. Observe that $(X \cup \{v_1\}, X \cup \{v_2\}, \dots, X \cup \{v_p\})$ is a path decomposition of G with width $|X| = n - \alpha(G) = \delta(G)$. Thus $\text{pw}(G) \leq \delta(G)$. \square

Lemma 8.3. *If H is a minor of a complete multipartite graph G , then H can be obtained from G by a sequence of edge contractions, vertex deletions, and edge deletions, such that each operation does not increase the minimum degree, connectivity, treewidth, or pathwidth.*

Proof. Every minor of a graph can be obtained by a sequence of edge contractions and vertex deletions, followed by a sequence of edge deletions. Contracting an edge or deleting a vertex in a complete multipartite graph produces another complete multipartite graph. Edge deletions do not increase the minimum degree, connectivity, treewidth, or pathwidth. Thus by Lemma 8.2, it suffices to prove that edge contractions and vertex deletions in complete multipartite graphs do not increase the minimum degree.

Say $G = K_{a_1, \dots, a_p}$ has n vertices. Then G has minimum degree $n - \max_i a_i$. Let G' be the graph obtained from G by contracting an edge. Then G' is a complete multipartite graph K_{1, a'_1, \dots, a'_p} with $n - 1$ vertices, where $a_i - 1 \leq a'_i \leq a_i$. Thus

$$\delta(G') = n - 1 - \max_i a'_i \leq n - 1 - \max_i (a_i - 1) = n - \max_i a_i = \delta(G) .$$

Now let G' be the graph obtained from G by deleting a vertex. Then G' is a complete multipartite graph $K_{a'_1, \dots, a'_p}$ with $n - 1$ vertices, where $a_i - 1 \leq a'_i \leq a_i$. By the same argument as before, $\delta(G') \leq \delta(G)$. \square

We now state and prove our first characterisation.

Theorem 8.4. *For all $k \geq 1$, the following are equivalent for a complete multipartite graph G :*

- (a) $G \in \widehat{\mathcal{C}}_k$
- (b) $G \in \widehat{\mathcal{D}}_k$
- (c) for some $b \geq a \geq 1$ and $p \geq 2$ such that $k + 1 = a + (p - 1)b$,

$$G = K_{\underbrace{a, b, \dots, b}_p},$$

and if $p = 2$ then $a = b$.

Proof. (b) \implies (a): Say $G \in \widehat{\mathcal{D}}_k$. By Lemma 2.1, $\delta(G) = k + 1$. By Lemma 8.2, $\kappa(G) = k + 1$. By (2), $G \in \widehat{\mathcal{C}}_k$.

(a) \implies (c): Consider a complete multipartite graph $G \in \widehat{\mathcal{C}}_k$. Thus $\kappa(G) \geq k + 1$ by Lemma 3.1. If $\kappa(G) \geq k + 2$ then $\kappa(G - e) \geq k + 1$ for any edge e of G , implying $G \notin \widehat{\mathcal{C}}_k$ by Lemma 3.1. Now assume that $\kappa(G) = k + 1$. Thus $\delta(G) = k + 1$ by Lemma 8.2.

Suppose on the contrary that adjacent vertices v and w in G both have degree at least $k + 2$. By Lemma 8.1, $G - vw$ is k -connected, implying $G \notin \widehat{\mathcal{C}}_k$. This contradiction proves that no two high-degree vertices in G are adjacent. If two vertices in a complete multipartite graph have distinct degrees, then they are adjacent. Thus the high-degree vertices in G have the same degree, and the vertices of G have at most two distinct degrees. Since the degree of each vertex v equals $|V(G)|$ minus the number of vertices in the colour class that contains v , the colour classes of G have at most two distinct sizes. Hence for some $b \geq a \geq 1$ and $p, q \geq 1$,

$$G = K_{\underbrace{a, \dots, a}_q, \underbrace{b, \dots, b}_p}.$$

Hence $\kappa(G) = aq + b(p - 1) = k + 1$. If $a = b$ then, taking $q = 1$, we are done. Now assume that $a < b$. Thus $q = 1$ as otherwise two high-degree vertices are adjacent. Thus

$$G = K_{\underbrace{a, b, \dots, b}_p}.$$

Suppose on the contrary that $p = 1$. Then $G = K_{a, b}$ and $\kappa(G) = a$. Contracting one edge in G gives $K_{1, a-1, b-1}$, which by Lemma 8.2 also has connectivity a , implying $G \notin \widehat{\mathcal{C}}_k$. This contradiction proves that $p \geq 2$.

Now suppose that $p = 2$. Then $G = K_{a, b, b}$ and $\kappa(G) = a + b$. Contracting one edge gives $K_{1, a, b-1, b-1}$, which by Lemma 8.2 also has connectivity $a + b$ (since $a < b$), implying $G \notin \widehat{\mathcal{C}}_k$. This contradiction proves that if $p = 2$ then $a = b$.

(c) \implies (b) Let

$$G = K_{\underbrace{a, b, \dots, b}_p},$$

for some $b \geq a \geq 1$ and $p \geq 2$, such that $k + 1 = a + (p - 1)b$ and if $p = 2$ then $a = b$. Thus G has minimum degree $k + 1$ by Lemma 8.2. Suppose on the contrary that $G \notin \widehat{\mathcal{D}}_k$. By Lemma 3.1, G has a proper minor H with $\delta(H) \geq k + 1$. By Lemma 8.3, every minor of G in the sequence from G to H has minimum degree at most $k + 1$. Thus we can assume that H was obtained from G by a single edge contraction, a vertex deletion, or an edge deletion. In each case we prove that $\delta(H) \leq k$, which is the desired contradiction.

First suppose that H is obtained from G by an edge contraction. Then

$$(i) H = K_{1, a-1, \underbrace{b, \dots, b}_{p-1}} \quad \text{or} \quad (ii) H = K_{1, a, b-1, \underbrace{b, \dots, b}_{p-2}}.$$

In case (i), $\delta(H) = 1 + (a - 1) + (b - 1) + (p - 2)b = k$. In case (ii) with $p \geq 3$, $\delta(H) = 1 + a + 2(b - 1) + (p - 3)b = k$. Now consider case (ii) with $p = 2$. By assumption, $a = b$. Thus $H = K_{1, a, a-1, a-1}$ has minimum degree $1 + 2(a - 1) = k$.

Now suppose that H is obtained from G by a vertex deletion. Then

$$(i) H = K_{\underbrace{a-1, b, \dots, b}_p} \quad \text{or} \quad (ii) H = K_{a, b-1, \underbrace{b, \dots, b}_{p-1}}.$$

In case (i), $\delta(H) = (a - 1) + (p - 1)b = k$. In case (ii), $\delta(H) = a + (b - 1) + (p - 2)b = k$ (since $p \geq 2$).

In G , every edge is incident to a vertex of degree $k + 1$. Thus, if H is obtained from G by an edge deletion, then $\delta(H) \leq k$. \square

The remainder of this section is devoted to characterising the complete multipartite graphs in $\widehat{\mathcal{T}}_k$ and in $\widehat{\mathcal{P}}_k$. We start with a lemma about independent sets in complete multipartite graphs.

Lemma 8.5. *For every edge vw in a complete multipartite graph G , every independent set in $G - vw$ is either $\{v, w\}$ or is also independent in G . Thus if $\alpha(G) \geq 2$ (that is, G is not a complete graph) then $\alpha(G - vw) = \alpha(G)$.*

Proof. Let $G' := G - vw$. Let I be an independent set in G' that is not independent in G . Thus both v and w are in I . Let S be the colour class containing v . Every vertex not in $S \cup \{w\}$ is adjacent to v in G' . Thus $I \subseteq S \cup \{w\}$. Every vertex in $S - \{v\}$ is adjacent to w in G' . Thus $I := \{v, w\}$. Hence every independent set in G' is either $\{v, w\}$ or is also independent in G . Thus $\alpha(G') = \alpha(G)$ whenever $\alpha(G) \geq 2$. \square

To prove lower bounds on treewidth we use the following idea. Let G be a graph. Two subgraphs X and Y of G *touch* if $X \cap Y \neq \emptyset$ or there is an edge of G between X and Y . A *bramble* in G is a set of pairwise touching connected subgraphs. The subgraphs are called *bramble elements*. A set S of vertices in G is a *hitting set* of a bramble \mathcal{B} if S

intersects every element of \mathcal{B} . The *order* of \mathcal{B} is the minimum size of a hitting set. The following ‘Treewidth Duality Theorem’ shows the intimate relationship between treewidth and brambles.

Theorem 8.6 (Seymour and Thomas [26]). *A graph G has treewidth at least k if and only if G contains a bramble of order at least $k + 1$.*

For example, say G is a complete multipartite graph on n vertices. Let S be a set of vertices in G , one from each colour class; that is, S is a maximum clique in G . Then it is easily seen that $\mathcal{B} := E(G) \cup S$ is a bramble of order $n - \alpha(G) + 1$, and thus $\text{tw}(G) \geq n - \alpha(G)$ by Theorem 8.6 (confirming Lemma 8.2). The next two lemmas give circumstances when an edge can be deleted from a complete multipartite graph without decreasing the treewidth.

Lemma 8.7. *Let G be a complete multipartite graph with $\alpha(G) \geq 3$, such that at least two colour classes contain at least two vertices. Let vw be an edge, where both v and w are in colour classes that contain at least two vertices. Then $\text{tw}(G - vw) = \text{tw}(G)$.*

Proof. Say G has n vertices. Let $G' := G - vw$. By Lemmas 8.2 and 8.5, $\text{tw}(G) = n - \alpha(G) = n - \alpha(G')$. Clearly $\text{tw}(G') \leq \text{tw}(G)$. Thus it suffices to prove that $\text{tw}(G') \geq n - \alpha(G')$.

Since v and w are in colour classes that contain at least two vertices, there is a set S of vertices, such that both v and w are not in S , and each colour class has exactly one vertex in S . Thus S is a maximum clique in G and in G' . Let $\mathcal{B} := E(G') \cup S$.

We now prove that \mathcal{B} is a bramble in G' . Each element of \mathcal{B} induces a connected subgraph in G' . Every pair of vertices in S are adjacent. Say $x \in S$ and $pq \in E(G')$. Since p and q have distinct colours, x is coloured differently from p or q , and thus x is adjacent to p or q (since $x \neq v$ and $x \neq w$). Hence x touches pq . Say $pq \in E(G')$ and $rs \in E(G')$. If $\{p, q\} \cap \{r, s\} \neq \emptyset$ then pq and rs touch. So assume that p, q, r, s are distinct. Thus there are at least two edges in G between $\{p, q\}$ and $\{r, s\}$, one of which is not vw . Hence pq touches rs . Therefore \mathcal{B} is a bramble in G' .

Let H be a minimum hitting set of \mathcal{B} . If $|H| \geq n - \alpha(G') + 1$, then \mathcal{B} has order at least $n - \alpha(G') + 1$, implying $\text{tw}(G') \geq n - \alpha(G')$ by Theorem 8.6, and we are done. Now assume that $|H| \leq n - \alpha(G')$.

Since every edge of G' is in \mathcal{B} , H is a vertex cover of G' , and $V(G') - H$ is an independent set of G' . Thus $n - |H| \leq \alpha(G')$. Hence $|H| = n - \alpha(G')$, and $V(G') - H$ is a maximum independent set of G' . By Lemma 8.5, every independent set of G' is $\{v, w\}$ or is an independent set of G . Since $\alpha(G') \geq 3$, $\{v, w\}$ is not a maximum independent set. Hence $V(G') - H$ is a maximum independent set of G . That is, $V(G) - H$ is a colour class in G , which implies that H does not contain one vertex in S , and H is not a hitting set of \mathcal{B} . This is the desired contradiction. \square

Lemma 8.8. *Let G be a complete multipartite graph with $\alpha(G) \geq 2$, and at least one singleton colour class. Let vw be an edge, where v is in a singleton colour class, and w is in a colour classes that contains at least two vertices. Then $\text{tw}(G - vw) = \text{tw}(G)$.*

Proof. Say G has n vertices. Let $G' := G - vw$. By Lemmas 8.2 and 8.5, $\text{tw}(G) = n - \alpha(G) = n - \alpha(G')$. Clearly $\text{tw}(G') \leq \text{tw}(G)$. Thus it suffices to prove that $\text{tw}(G') \geq n - \alpha(G')$.

By assumption, there is a set S of vertices, such that $w \notin S$, and every colour class has exactly one vertex in S . Thus $v \in S$. Note that S is a maximum clique in G and in G' . Let $\mathcal{B} := E(G') \cup S$.

We now prove that \mathcal{B} is a bramble in G' . Each element of \mathcal{B} induces a connected subgraph in G' . Every pair of vertices in S are adjacent. Consider $v \in S$ and $pq \in E(G')$. Since v is in a singleton colour class, v is adjacent to both p and q in G , and thus v is adjacent to p or q in G' . Hence v touches pq . Now consider $x \in S - \{v\}$ and $pq \in E(G')$. Since p and q have distinct colours, x is coloured differently from p or q , and thus x is adjacent to p or q (since $x \neq v$ and $x \neq w$). Hence x touches pq . Finally consider two edges $pq \in E(G')$ and $rs \in E(G')$. If $\{p, q\} \cap \{r, s\} \neq \emptyset$ then pq and rs touch. So assume that p, q, r, s are distinct. Thus there are at least two edges in G between $\{p, q\}$ and $\{r, s\}$, one of which is not vw . Hence pq touches rs . Therefore \mathcal{B} is a bramble in G' .

Let H be a minimum hitting set of \mathcal{B} . If $|H| \geq n - \alpha(G') + 1$, then \mathcal{B} has order at least $n - \alpha(G') + 1$, implying $\text{tw}(G') \geq n - \alpha(G')$ by Theorem 8.6, and we are done. Now assume that $|H| \leq n - \alpha(G')$.

Since every edge of G' is in \mathcal{B} , H is a vertex cover of G' , and $V(G') - H$ is an independent set of G' . Thus $n - |H| \leq \alpha(G')$. Hence $|H| = n - \alpha(G')$, and $V(G') - H$ is a maximum independent set of G' . By Lemma 8.5, every independent set of G' is $\{v, w\}$ or is an independent set of G . If $V(G') - H = \{v, w\}$ then H does not contain v , and H is not a hitting set of \mathcal{B} , which is a contradiction. Otherwise, $V(G) - H$ is a maximum independent set of G . That is, $V(G) - H$ is a colour class in G , which implies that H does not contain some vertex in S , and H is not a hitting set of \mathcal{B} . This is the desired contradiction. \square

Theorem 8.9. *For all $k \geq 1$, the following are equivalent for a complete multipartite graph G :*

- (a) $G \in \widehat{\mathcal{T}}_k$
- (b) $G \in \widehat{\mathcal{P}}_k$
- (c) $G = K_{k+2}$, or $k \geq 3$ is odd and $G = K_{\underbrace{2, \dots, 2}_{(k+3)/2}}$.

Proof. (b) \implies (a): Say $G \in \widehat{\mathcal{P}}_k$. By Lemma 3.1, $\text{pw}(G) = k + 1$. By Lemma 8.2, $\text{tw}(G) = k + 1$. By (4), $G \in \widehat{\mathcal{T}}_k$.

(a) \implies (c): Say $G \in \widehat{\mathcal{T}}_k$. By Lemma 3.1, $\text{tw}(G) \geq k + 1$. If $\text{tw}(G) \geq k + 2$ then $\text{tw}(G - v) \geq k + 1$ for any vertex v of G , implying $G \notin \widehat{\mathcal{T}}_k$ by Lemma 3.1. Now assume that $\text{tw}(G) = k + 1$. Thus $\delta(G) = k + 1$ by Lemma 8.2, and $G \in \widehat{\mathcal{D}}_k$ by (3). By Theorem 8.4,

$$G = K_{\underbrace{a, b, \dots, b}_p},$$

for some $b \geq a \geq 1$ and $p \geq 2$, such that $k + 1 = a + (p - 1)b$ and if $p = 2$ then $a = b$.

Case. $b = 1$: Then $a = 1$ and $G = K_{k+2}$, and we are done.

Case. $b = 2$: Then $k + 3 = a + 2p$. If $a = 1$, then by Lemma 8.8, $\text{tw}(G - e) = \text{tw}(G)$ for some edge e of G , implying $G \notin \mathcal{T}_k$ by Lemma 3.1. Otherwise $a = 2$. Thus $k = 2p - 1$ is odd, and $k \geq 3$ since $p \geq 2$. Hence

$$G = K_{\underbrace{2, \dots, 2}_{(k+3)/2}}.$$

Case. $b \geq 3$: Then $\alpha(G) \geq 3$. Since $p \geq 2$, there are at least two colour class that contain at least two vertices, and by Lemma 8.7, $\text{tw}(G - e) = \text{tw}(G)$ for some edge e of G , implying $G \notin \mathcal{T}_k$ by Lemma 3.1.

(c) \implies (b): If $G = K_{k+2}$ then $G \in \widehat{\mathcal{P}}_k$ by Lemma 3.1. Now assume that $k \geq 3$ is odd and

$$G = K_{\underbrace{2, \dots, 2}_{(k+3)/2}}.$$

Thus $\text{pw}(G) = k + 1$ by Lemma 8.2. Suppose on the contrary that $G \notin \widehat{\mathcal{P}}_k$. By Lemma 3.1, G has a proper minor H with $\text{pw}(H) \geq k + 1$. By Lemma 8.3, every minor of G in the sequence from G to H has pathwidth at most $k + 1$. Thus we can assume that H was obtained from G by a single edge contraction, a vertex deletion, or an edge deletion. Since an edge contraction or a vertex deletion produce another complete multipartite graph, and the minimum degree of a complete multipartite graph equals its pathwidth (Lemma 8.2), the same proof used in Theorem 8.4 shows that $\text{pw}(H) \leq k$. Now assume that $H = G - vw$ for some edge vw of G . Let x be the other vertex in the colour class that contains v . Let y be the other vertex in the colour class that contains w . Let $S := V(G) - \{v, w, x, y\}$. Then $(S \cup \{v, y\}, S \cup \{x, y\}, S \cup \{x, w\})$ is a path decomposition of H with width k , which is the desired contradiction. \square

Open Problem 8.10. Complete multipartite graphs have diameter 2. Are there generalisations of Theorems 8.4 and 8.9 for all diameter-2 graphs in $\widehat{\mathcal{D}}_k$ or in $\widehat{\mathcal{T}}_k$?

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A Graphs with Minimum Degree Four

In this appendix we prove the following result.

Theorem A.1. *Every graph with minimum degree at least 4 contain a 4-connected minor.*

The following stronger result enables an inductive proof of Theorem A.1.

Lemma A.2. *Let G be a graph with at least 5 vertices, such that the vertices of degree at most 3 induce a clique. Then G contains a 4-connected minor.*

Proof. Let G be a counterexample with the minimum number of vertices. Let $K = \{v_1, \dots, v_{|K|}\}$ denote the (possibly empty) clique of vertices of degree at most 3.

In each case below we exhibit a proper minor G' of G , for which it is easy to verify that the vertices of degree at most 3 induce a clique. Moreover, $|V(G')| \geq 5$ since there is a vertex of degree at least 4 in G , whose degree does not decrease in G' . Thus G' satisfies the conditions of the lemma, which contradicts the minimality of G .

If e is an edge incident to a vertex of degree at most 2, then let $G' := G/e$. Now assume that $\delta(G) \geq 3$.

Let S be a minimal separator in G , and let $\{G_1, G_2\}$ be the corresponding separation, so that $S = V(G_1 \cap G_2)$. Without loss of generality, $K \subseteq G_1$. If $|S| = 1$, then let $G' := G_2$. If $|S| = 2$, say $S = \{s_1, s_2\}$, then there exists an s_1 – s_2 path in G_1 , and so $G' := G_2 + s_1 s_2$ is the desired minor.

Thus G is 3-connected and each vertex in K has degree 3. Let $N(K)$ denote the subgraph induced by the neighbours of K .

First suppose that $|K| = 2$. Both v_1 and v_2 have at least two neighbours in $N(K)$. If $|N(K)| \geq 3$, then at most one vertex is adjacent to both v_1 and v_2 . Let $G' := G/v_1 v_2$. If, on the other hand, $N(K) = \{u_1, u_2\}$, then let G' be obtained from G by contracting the triangle $v_1 v_2 u_1$. Since G has a vertex of degree at least 4 other than u_1, u_2 , so does G' .

If $|K| = 1$ or $|K| = 3$ then $|N(K)| = 3$. Every vertex of $N(K)$ is adjacent to exactly one vertex of K , so G has more than $|K| + |N(K)|$ vertices. If $N(K)$ induces a clique, then let $G' := G - V(K)$. Otherwise let u_1 be a vertex whose degree in $N(K)$ is as

small as possible (is at most 1) and let v_1 be its unique neighbour in K . In this case, let $G' := G/u_1v_1$. Therefore we may assume that G is 3-connected with $\delta(G) \geq 4$.

Suppose that G contains a 3-separation $\{G_1, G_2\}$ with separator $S = \{s_1, s_2, s_3\} = V(G_1 \cap G_2)$. Consider the subgraph G_1 . Each vertex in S has degree at least 1 in G_1 . Now $V(G_1 - S) \neq \emptyset$ and every vertex in $G_1 - S$ has degree at least 4 in G_1 . A forest that contains a vertex of degree at least 4 has at least 4 leaves. Thus G_1 is not a forest. Hence G_1 contains a cycle C . By Menger's Theorem, there are three disjoint C - S paths in G_1 . By contracting C together with these three paths to a triangle on S , observe that $G' := G_2 + s_1s_2 + s_1s_3 + s_2s_3$ is the desired minor. Hence G has no 3-separation and is thus 4-connected. \square

For completeness we include a proof of the following theorem of Halin and Jung [17] based on classical results by Wagner, Whitney and Tutte.

Theorem A.3 ([17]). *Every 4-connected graph contains K_5 or $K_{2,2,2}$ as a minor.*

Proof. Suppose that G is 4-connected and has no K_5 -minor. Thus G is planar by Wagner's characterisation of graphs with no K_5 -minor [29]. Fix a plane embedding of G . Let v be any vertex of G . Let w_1, w_2, w_3, w_4 be four of the neighbours of v in cyclic order around v . Let C be the facial cycle in the induced plane embedding of $G - v$, such that the interior of C contains v . Whitney [30] proved that every 3-connected planar graph has a unique plane embedding. Moreover, Tutte [28] proved that the faces of this embedding are exactly the induced nonseparating cycles. Since $G - v$ is 3-connected, each face in the induced plane embedding of $G - v$ is an induced nonseparating cycle. In particular, C is induced and nonseparating in $G - v$. Since C is separating, $(G - v) - C$ is connected. Since C is induced, each vertex w_i has exactly two neighbours in C , and at least one neighbour in $(G - v) - C$. Hence, contracting $(G - v) - C$ to a single vertex, and contracting C to the 4-cycle (w_1, w_2, w_3, w_4) produces a $K_{2,2,2}$ -minor in G . \square

Note that Theorem A.3 can also be concluded from a theorem of Maharry [23], who proved that every 4-connected graph with no $K_{2,2,2}$ minor is isomorphic to the square of an odd cycle, which is easily seen to contain a K_5 -minor. Theorems A.1 and A.3 imply:

Corollary A.4. *Every graph with minimum degree at least 4 contains K_5 or $K_{2,2,2}$ as a minor.*