



Geometric thickness in a grid

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Abstract

The geometric thickness of a graph is the minimum number of layers such that the graph can be drawn in the plane with edges as straight-line segments, and with each edge assigned to a layer so that no two edges in the same layer cross. We consider a variation on this theme in which each edge is allowed one bend. We prove that the vertices of an n -vertex m -edge graph can be positioned in a $\lceil\sqrt{n}\rceil \times \lceil\sqrt{n}\rceil$ grid and the edges assigned to $\mathcal{O}(\sqrt{m})$ layers, so that each edge is drawn with at most one bend and no two edges on the same layer cross. The proof is a 2-dimensional generalisation of a theorem of Malitz (J. Algorithms 17(1) (1994) 71–84) on book embeddings. We obtain a Las Vegas algorithm to compute the drawing in $\mathcal{O}(m \log^3 n \log \log n)$ time (with high probability).

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1. Introduction

Let $G=(V,E)$ be a simple graph with $n=|V|$ vertices, $m=|E|$ edges, maximum degree Δ , and genus γ . The *geometric thickness* of G , denoted by $\bar{\theta}(G)$, is the minimum number of layers such that G can be drawn in the plane with edges as straight-line segments, and with each edge assigned to a layer so that no two edges in the same layer cross. Geometric thickness was first introduced under the name of *real linear thickness* by Kainen [19], and has recently been studied by Dillencourt et al. [12]. Applications

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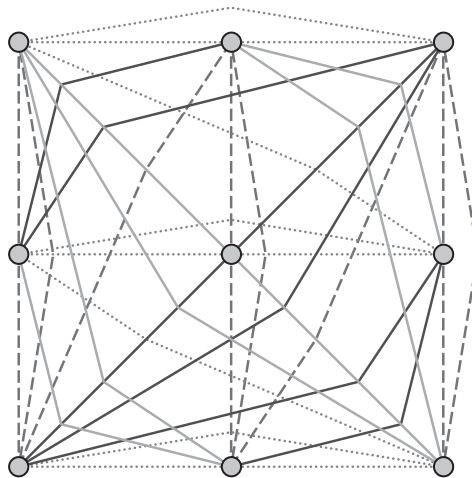


Fig. 1. A drawing of K_9 in the 3×3 grid with 4 layers and at most 1 bend per edge.

of geometric thickness include multilayer VLSI [1,2] and graph visualisation where layers correspond to colours in a drawing.

Geometric thickness is related to the (graph-theoretic) *thickness* of a graph, defined to be the minimum number of subgraphs in a partition of the edges into planar subgraphs, denoted by $\theta(G)$. See [24] for a survey of results concerning this well-studied parameter. A graph G has a drawing in the plane with an arbitrary set of pre-assigned vertex locations with edges assigned to $\theta(G)$ layers such that no two edges in the same layer cross [19]. Thus, the key difference between geometric thickness and graph-theoretic thickness is that geometric thickness requires the edges to be drawn as straight line-segments, whereas the graph-theoretic thickness allows edges to bend arbitrarily.

In this paper, we consider a variation of geometric thickness which lies between thickness and geometric thickness in which vertices are placed in a grid and each edge has at most one bend. In VLSI circuit design, each edge typically has a small number of bends (called *jogs*) (see [20, p. 222]). We are also interested in drawings with small area, which is an important consideration in VLSI and visualisation. To consistently measure the area of a drawing we assume a *vertex resolution rule*; that is, pairs of vertices are at least unit-distance apart.

Our main result is a 2-dimensional generalisation of the randomized algorithm of Malitz [22] for producing book embeddings (see Section 1.1). We prove that the vertices of a graph can be positioned in a $\lceil \sqrt{n} \rceil \times \lceil \sqrt{n} \rceil$ grid, such that the edges can be partitioned into $\mathcal{O}(\sqrt{m})$ layers, with each edge drawn with at most one bend, so that no two edges in the same layer cross. We obtain a Las Vegas algorithm to produce the drawing in $\mathcal{O}(m \log^3 n \log \log n)$ time (with high probability). An example of the drawings produced by our algorithm is shown in Fig. 1. As a by-product our algorithm constructs a drawing of a book embedding in the plane with $\mathcal{O}(n)$ area and

with each edge having at most one bend, such that edges in the same page do not cross.

In the following section we review related work. Section 2 introduces notation and some preliminary results required to prove the main result in Section 3.

1.1. Related work

Since every planar graph has a drawing in the plane with straight-line edges [15,27], the graphs with geometric thickness 1 are precisely the planar graphs. Graphs with geometric thickness 2, called *doubly linear* graphs, are studied in [6,18]. Hutchinson et al. [18] prove that every doubly linear graph has at most $6n - 18$ edges, and present doubly linear graphs with $6n - 20$ edges for all $n \geq 8$.

Dillencourt et al. [12] establish lower and upper bounds for the geometric thickness of complete and complete bipartite graphs. It is shown that $\lceil n+1/5.646 \rceil \leq \bar{\theta}(K_n) \leq \lceil n/4 \rceil$. Note that their construction has $\mathcal{O}(n^6)$ area under the vertex resolution rule (D. Eppstein, personal communication). Since $\theta(K_n) = n/6 + \mathcal{O}(1)$ [3,5,4,23], $\bar{\theta}(K_n) > \theta(K_n)$ for large enough n . On the other hand, $\bar{\theta}(K_{a,b}) = \theta(K_{a,b})$ when $a \gg b$ [12].

Another parameter closely related to geometric thickness is that of book thickness, introduced independently by Cottafava and D'Antona [9] and Bernhart and Kainen [7]. A *book* consists of a line in 3-space, called the *spine*, and some number of *pages*, each a half-plane with the spine as boundary. A *book embedding* of a graph consists of a linear ordering of the vertices, called the *spine ordering*, along the spine of a book and an assignment of edges to pages so that edges assigned to the same page can be drawn on that page without crossings. The *book thickness* of a graph G , denoted by $\text{bt}(G)$, is the minimum number of pages in a book embedding of G . Book thickness has also been called *pagenumber* and *stacknumber* in the literature. Book thickness is equivalent to the geometric thickness with the additional requirement that each layer induces an outerplanar subgraph; hence $\bar{\theta}(G) \leq \text{bt}(G)$. To see this, position the vertices on a circle in the plane according to their order along the spine, and draw the edges as straight line-segments [7, Lemma 2.1]. Note that this construction has $\mathcal{O}(n^2)$ area under the vertex resolution rule.

Upper bounds for the thickness of an arbitrary graph G include $\theta(G) \leq \lfloor \sqrt{m/3} + 3/2 \rfloor$ by Dean et al. [11], $\theta(G) \leq 6 + \sqrt{2\gamma - 2}$ by Dean and Hutchinson [10], and $\theta(G) \leq \lceil \Delta/2 \rceil$ by Halton [16]. Of these three upper bounds, the first two are asymptotically matched by the book thickness. In particular, Malitz [21] proved that $\text{bt}(G) \in \mathcal{O}(\sqrt{\gamma})$, and since $\gamma \leq m$, $\text{bt}(G) \in \mathcal{O}(\sqrt{m})$, a result proved independently by Malitz [22]. It is an open problem to decide if the third of these upper bounds is asymptotically matched by the geometric thickness; that is, is $\bar{\theta}(G) \in \mathcal{O}(\Delta)$?

2. Drawings in a grid

The $N \times N$ *grid* is the set of *gridpoints* $\{(x, y) \in \mathbb{N} \times \mathbb{N} : 1 \leq x, y \leq N\}$ in the plane. The x_0 -*column* of the $N \times N$ grid is the set of gridpoints $\{(x_0, y) : 1 \leq y \leq N\}$. The y_0 -*row* of the $N \times N$ grid is the set of gridpoints $\{(x, y_0) : 1 \leq x \leq N\}$. Assuming

N is a power of 2, for each $a = 1, 2, \dots, \log N$, partition the gridpoints into 2^a sets of $N/2^a$ consecutive columns, called the *vertical a -strips*, which we alternately label L, R, L, R , etc. from left to right. (Unless stated otherwise, all logarithms are base 2.) For $b = 1, 2, \dots, \log N$, partition the gridpoints into 2^b sets of $N/2^b$ consecutive rows, called the *horizontal b -strips*, which we alternately label D, U, D, U , etc. from bottom to top. The intersection of a vertical a -strip and a horizontal b -strip is called an (a, b) -region. Note that a $(\log N, b)$ -region is contained within a single column, and an $(a, \log N)$ -region is contained within a single row.

Each (a, b) -region P labelled L and D is *coupled* with the (a, b) -region Q labelled R and U immediately above and to the right of P ; (P, Q) is called a *positive* pair of regions. Similarly, each (a, b) -region P labelled R and D is *coupled* with the (a, b) -region Q labelled L and U immediately above and to the left of P ; (P, Q) is called a *negative* pair of regions. In addition, for each b , $1 \leq b \leq \log N$, each $(\log N, b)$ -region P labelled D is *coupled* with the $(\log N, b)$ -region Q labelled U immediately above it; (P, Q) is called a *column* pair of regions. Similarly, for each a , $1 \leq a \leq \log N$, each $(a, \log N)$ -region P labelled L is *coupled* with the $(a, \log N)$ -region Q labelled R immediately to the right it; (P, Q) is called a *row* pair of regions.

An $N \times N$ *grid layout* of a graph G (with N not necessarily a power of 2) is a 1-1 mapping π from the vertices of G to the gridpoints of an $N \times N$ grid. Denote $\pi(v)$ by $(X_\pi(v), Y_\pi(v))$ for each vertex v of G . Note that the $N \times N$ grid layouts of an N^2 -vertex graph correspond to the permutations of the vertices. In a grid layout of a graph G , for each positive (respectively, negative, column, row) pair of regions (P, Q) the 2-coloured bipartite subgraph $H = (V_P, V_Q, E_H)$ consisting of edges vw of G with $\pi(v) \in P$ and $\pi(w) \in Q$ is called a (*regional*) *positive* (respectively, *negative*, *column*, *row*) *subgraph*. Clearly each edge of G appears in precisely one regional subgraph. The set of edges in some positive (respectively, negative, row, column) subgraph are denoted by E^+ (E^- , E^r , E^c). The regional subgraph corresponding to a coupled pair of (a, b) -regions is called an (a, b) -*subgraph*. See Fig. 2 for an illustration of these concepts in the case of an 8×8 grid layout.

If $H = (V_P, V_Q, E_H)$ is the regional subgraph containing the edge e then $P(e)$ refers to the end-vertex of e in V_P , and $Q(e)$ refers to the end-vertex of e in V_Q . We consider e to be oriented from $P(e)$ to $Q(e)$. An edge vw oriented from v to w is denoted by \vec{vw} .

2.1. Edges in row and column subgraphs

We now describe how to draw and assign layers for edges in E^r . The case for edges in E^c can be easily inferred. The method essentially describes how to draw a book embedding with at most one bend per edge. For each y_0 , $1 \leq y_0 \leq N$, let $E_{y_0}^r = \{\vec{vw} \in E^r : Y_\pi(v) = Y_\pi(w) = y_0\}$. As illustrated in Fig. 3, an edge $\vec{vw} \in E_{y_0}^r$ is drawn with a bend at

$$\left(\frac{1}{2}(X_\pi(w) + X_\pi(v)), y_0 + \left(\frac{1}{N}(X_\pi(w) - X_\pi(v) - 1) \right)^2 \right).$$

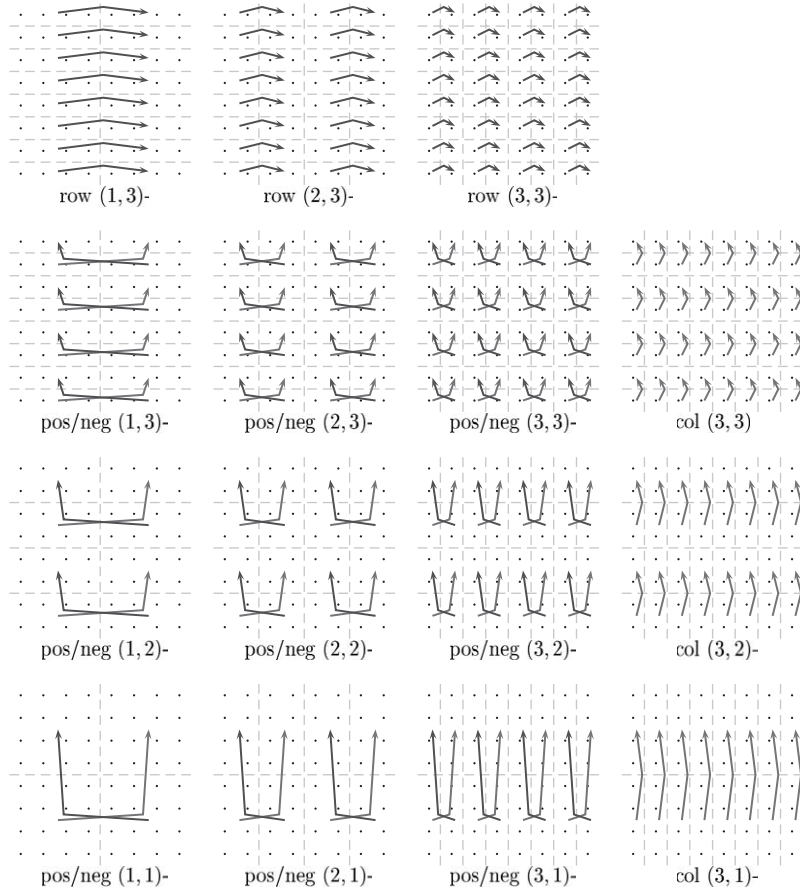


Fig. 2. (a, b) -subgraphs in an 8×8 grid layout.

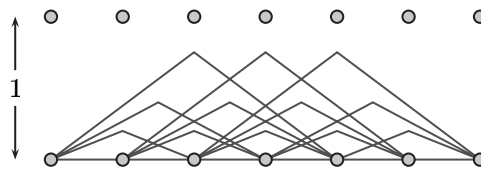


Fig. 3. 1-bend edges in row subgraphs.

By considering similar triangles, it is easily verified that two edges vw and xy in $E_{y_0}^r$ cross if and only if $X_\pi(v) < X_\pi(x) < X_\pi(w) < X_\pi(y)$ or $X_\pi(x) < X_\pi(v) < X_\pi(y) < X_\pi(w)$. A set of edges contained in $E_{y_0}^r$ is said to be *completely crossing* if edges are pairwise crossing. Such a subgraph is a *matching*; that is, every vertex has degree one. The following lemma is equivalent to Lemma 2.2 of Malitz [22].

Lemma 1. *Let $H = (V_P, V_Q, E_H)$ be a row or column subgraph in a grid layout π of a graph G . If H has $n = |V_P \cup V_Q|$ vertices and $m = |E|$ edges, and at most k edges in H are completely crossing, then the edges of H can be partitioned into k layers in $\mathcal{O}(m \log \log n)$ time such that with the edges drawn as described above, no two edges in a single layer cross.*

Proof. We prove the result for a row subgraph. The proof for a column subgraph is analogous. Define a partial ordering \preceq on E_H as follows. Let

$$e_1 \preceq e_2 \stackrel{\text{def}}{=} (X_\pi(P(e_2)) \leq X_\pi(P(e_1)) \quad \text{and} \quad X_\pi(Q(e_1)) \leq X_\pi(Q(e_2))).$$

It is a simple exercise to check that \preceq is reflexive, transitive and antisymmetric, and thus is a partial order. Two edges are incomparable under \preceq if and only if they cross. Thus an antichain is a completely crossing set of edges, and a chain is a set of pairwise non-crossing edges. By Dilworth's Theorem [13] there is a decomposition of the edges of H into k chains where k is the size of the largest antichain. That is, there is a partition of the edges of H into k layers such that no two edges in a single layer cross, where k is the maximum number of completely crossing edges in H . The time complexity can be achieved using a dual form of the algorithm by Heath and Rosenberg [17, Theorem 2.3] (see [26, Lemma 2.1]). \square

The following result is an easy consequence of the above construction.

Corollary 2. *A graph $G = (V, E)$ has a drawing with $\mathcal{O}(n)$ area such that each edge has at most one bend, and with the edges partitioned into $\text{bt}(G)$ layers, so that no two edges in a single layer cross.*

2.2. Edges in positive and negative subgraphs

We now describe how to draw and assign layers for edges in E^+ and E^- . For each x_0, y_0 , $1 \leq x_0, y_0 \leq N$, let

$$E_{x_0, y_0}^+ = \{\vec{vw} \in E^+ : X_\pi(v) < X_\pi(w) = x_0 \text{ and } y_0 = Y_\pi(v) < Y_\pi(w)\},$$

$$E_{x_0, y_0}^- = \{\vec{vw} \in E^- : y_0 = Y_\pi(v) < Y_\pi(w) \text{ and } X_\pi(v) > X_\pi(w) = x_0\}.$$

Thus E_{x_0, y_0}^+ (respectively, E_{x_0, y_0}^-) consists of edges in positive (negative) subgraphs oriented from a vertex in the y_0 -row to a vertex in the x_0 -column. As illustrated in Fig. 4, for each E_{x_0, y_0}^\pm , assign evenly spaced bends along the line-segment

$$(x_0 \mp \frac{1}{2}, y_0 + \frac{1}{2}) \rightarrow \left(x_0 \mp \frac{1}{2} \mp \frac{1}{\sqrt{2N}}, y_0 + \frac{1}{2} + \frac{1}{\sqrt{2N}} \right)$$

to the edges $\vec{vw} \in E_{x_0, y_0}^\pm$ in increasing order of $|x_0 - X_\pi(v)| + |y_0 - Y_\pi(w)|$ (breaking ties arbitrarily).

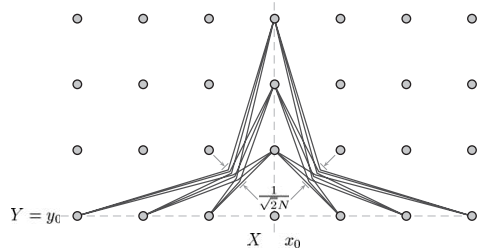


Fig. 4. Routing edges in $E^+_{x_0,y_0}$ and $E^-_{x_0,y_0}$.

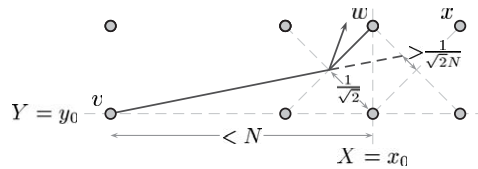


Fig. 5. Adjacent edges in a positive subgraph do not cross.

To enable characterisations of when edges in E^+ or E^- cross, we define the following total orderings on gridpoints:

$$(x_1, y_1) <_{N,E} (x_2, y_2) \stackrel{\text{def}}{=} (y_1 < y_2) \text{ or } (y_1 = y_2 \text{ and } x_1 < x_2),$$

$$(x_1, y_1) <_{N,W} (x_2, y_2) \stackrel{\text{def}}{=} (y_1 < y_2) \text{ or } (y_1 = y_2 \text{ and } x_1 > x_2),$$

$$(x_1, y_1) <_{E,S} (x_2, y_2) \stackrel{\text{def}}{=} (x_1 < x_2) \text{ or } (x_1 = x_2 \text{ and } y_1 > y_2),$$

$$(x_1, y_1) <_{W,S} (x_2, y_2) \stackrel{\text{def}}{=} (x_1 > x_2) \text{ or } (x_1 = x_2 \text{ and } y_1 > y_2).$$

Lemma 3. *Let e_1 and e_2 be edges in a positive subgraph $H = (V_P, V_Q, E_H)$ in a grid layout of a graph G . If e_1 and e_2 are adjacent then e_1 and e_2 do not cross. If e_1 and e_2 are non-adjacent and $P(e_1) <_{N,W} P(e_2)$ then e_1 and e_2 cross if and only if $Q(e_1) <_{E,S} Q(e_2)$.*

Proof. We first prove that adjacent edges in H do not cross. Since bends are assigned to edges in each $E^+_{x_0,y_0}$ in non-decreasing order of $|x_0 - X_\pi(v)| + |y_0 - Y_\pi(w)|$, adjacent edges in the same $E^+_{x_0,y_0}$ do not cross. Let vw be an edge in $E^+_{x_0,y_0}$ with v in the y_0 -row and w in the x_0 -column. Consider the first segment of the edge from v to w extended to intersect the line segment containing the bends for edges in $E^+_{x_0+1,y_0}$, as illustrated in Fig. 5. By considering similar triangles, it is easily verified that the distance from this point of intersection to $(x_0 + \frac{1}{2}, y_0 + \frac{1}{2})$ is greater than $1/\sqrt{2N}$. Thus an edge $vx \in E^+_{x_0+i,y_0}$ for any $i \geq 1$ will not cross vw . By symmetry, edges in H incident to a vertex in an x_0 -column do not cross.

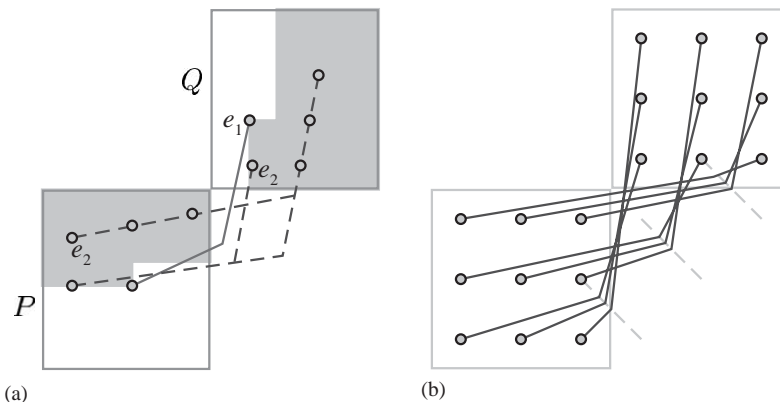


Fig. 6. In a positive subgraph: (a) crossing edges with $P(e_1) <_{N,W} P(e_2)$; and (b) a completely crossing matching.

Now suppose e_1 and e_2 are not adjacent and $P(e_1) <_{N,W} P(e_2)$. Fig. 6(a) shows the ways in which two edges e_1 and e_2 in H with $P(e_1) <_{N,W} P(e_2)$ can cross. The shaded region within P shows the possible locations for $P(e_2)$ relative to $P(e_1)$ such that $P(e_1) <_{N,W} P(e_2)$. The shaded region within Q shows where $Q(e_2)$ can be relative to $Q(e_1)$ so that e_1 and e_2 cross. Clearly e_1 and e_2 cross if and only if $Q(e_1) <_{E,S} Q(e_2)$. \square

A set of pairwise crossing edges in a positive or negative subgraph are said to be *completely crossing*. By Lemma 3, a completely crossing set of edges is a *matching*; that is, every vertex has degree one (see Fig. 6(b)). The following lemma for completely crossing matchings in a positive or negative subgraph is an analogue of Lemma 1.

Lemma 4. *Let $H = (V_P, V_Q, E_H)$ be a positive or negative subgraph in a grid layout π of a graph G . If H has $n = |V_P \cup V_Q|$ vertices and $m = |E_H|$ edges, and at most k edges are completely crossing, then the edges of H can be partitioned into k layers in $\mathcal{O}(m \log \log n)$ time, such that no two edges in a single layer cross, and with the edges drawn with one bend as described above.*

Proof. Suppose H is a positive subgraph. Define a partial order \preceq on E_H by

$$e_1 \preceq e_2 \stackrel{\text{def}}{=} P(e_1) \preceq_{N,W} P(e_2) \quad \text{and} \quad Q(e_2) \preceq_{E,S} Q(e_1).$$

It is a simple exercise to check that \preceq is reflexive, transitive and antisymmetric, and thus is a partial order. By Lemma 3, two edges are incomparable under \preceq if and only if they cross. Thus, as in Lemma 1, E_H can be partitioned in $\mathcal{O}(m \log \log n)$ time into k crossing-free layers, where k is the maximum number of completely crossing edges in H .

The case in which H is a negative subgraph is analogous. By symmetry adjacent edges in H do not cross. Let e_1 and e_2 be non-adjacent edges in a negative subgraph

$H = (V_P, V_Q, E_H)$ such that $P(e_1) <_{N,E} P(e_2)$. By a similar argument to that in Lemma 3, e_1 and e_2 cross if and only if $Q(e_1) <_{W,S} Q(e_2)$. Define a partial order on E_H by $e_1 \preceq e_2$ if $P(e_1) \leq_{N,E} P(e_2)$ and $Q(e_2) \leq_{W,S} Q(e_1)$. Again two edges are incomparable under \preceq if and only if they cross, and applying Dilworth's Theorem we obtain the desired partition. \square

Lemmas 1 and 4 together imply that for regional subgraphs there is a polynomial time algorithm to optimally assign layers to edges. The algorithm presented in the following section uses different sets of layers for edges in positive, negative, row and column subgraphs. Note that for each pair a, b ($1 \leq a, b \leq \log N$), the positive (a, b) -subgraphs can share the same set of layers, and similarly for negative, row and column subgraphs.

3. Main proof

We are now ready to prove the main result. Its proof is a 2-dimensional generalisation of Theorem 2.3 by Malitz [22] for book embeddings, which in turn is based on ideas from Theorem 4.7 by Chung et al. [8].

Theorem 5. *The vertices of a connected graph $G = (V, E)$ can be positioned in a $\lceil \sqrt{|V|} \rceil \times \lceil \sqrt{|V|} \rceil$ grid, such that the edges of G can be partitioned into $\mathcal{O}(\sqrt{|E|})$ layers, with each edge drawn with at most one bend, so that no two edges in the same layer cross.*

Proof. Let $m = |E|$ and $n = \lceil \sqrt{|V|} \rceil^2$. Add $n - |V|$ isolated vertices to G . Now G has n vertices and \sqrt{n} is an integer. Clearly $\lceil \sqrt{|V|} \rceil \leq \sqrt{2|V|}$ and thus $n \leq 2|V|$. Since G is connected, $|V| \leq m + 1 \leq 2m$ and $n \leq 4m$. A drawing of (the new) G in the $\sqrt{n} \times \sqrt{n}$ grid contains a drawing of the original G in a $\lceil \sqrt{|V|} \rceil \times \lceil \sqrt{|V|} \rceil$ grid. Let π be a random grid layout of G in the $\sqrt{n} \times \sqrt{n}$ grid. (Such grid layouts of G correspond to permutations of the vertices.)

Let $N = 2^{\lceil \log \sqrt{n} \rceil}$; that is, N is the minimum power of 2 no less than \sqrt{n} . Note that $N < 2^{1+\log \sqrt{n}} = 2\sqrt{n}$ and thus $n > (N^2/4)$. We consider the $\sqrt{n} \times \sqrt{n}$ grid layout π to be positioned at the bottom left corner of an $N \times N$ grid. Regions and regional subgraphs are then defined with respect to this larger grid, while vertices only lie in the smaller $\sqrt{n} \times \sqrt{n}$ grid.

An (a, b) -subgraph H with $a + b = j$ is said to be in the j -level. This definition has the effect of grouping regional subgraphs by the number of gridpoints in the corresponding regions, since an (a, b) -region has $N^2/2^{a+b}$ gridpoints.

We first count the number of positive or negative subgraphs in each j -level. For each a and b , $1 \leq a, b \leq \log N$, there are 2^{a+b-1} positive or negative (a, b) -subgraphs. For each j , there are at most $j - 1$ pairs (a, b) with $a + b = j$ and $1 \leq a, b \leq \log N$. Thus each j -level contains at most $(j - 1)2^{j-1}$ positive or negative subgraphs. Now we count the row and column subgraphs. With $b = \log N$, there are 2^{a+b-1} row (a, b) -subgraphs for each a , $1 \leq a \leq \log N$, and with $a = \log N$, there are 2^{a+b-1} column (a, b) -subgraphs

for each b , $1 \leq b \leq \log N$. Therefore each j -level contains at most $(j+1)2^{j-1}$ regional subgraphs.

For each j , $2 \leq j \leq 2 \log N$, let A_k^j be the event that there exists a 2-coloured matching M in G with k edges such that M is completely crossing and is contained in a j -level regional subgraph with respect to π . The probability that A_k^j occurs

$$\mathbf{P}\{A_k^j\} < \underbrace{\binom{m}{k} 2^k}_{(1)} \cdot \underbrace{(j+1)2^{j-1}}_{(2)} \cdot \underbrace{\left(\frac{N^2}{2^j}\right)^2 \frac{(k!)^2 (n-2k)!}{n!}}_{(3)} \cdot \underbrace{\frac{1}{k!}}_{(4)},$$

where

- (1) is an upper bound on the number of 2-coloured k -edge matchings $M = (M_P, M_Q, E_M)$,
- (2) is an upper bound on the number of j -level regional subgraphs $H = (V_P, V_Q, E_H)$,
- (3) is an upper bound on the probability that M is contained in H with $M_P \subseteq V_P$ and $M_Q \subseteq V_Q$, and
- (4) is the probability that M is completely crossing in π , given fixed gridpoints for the end-vertices of M .

Since $\binom{a}{b} \leq a^b/b!$,

$$\mathbf{P}\{A_k^j\} < (2m)^k \cdot (j+1)2^{j-1} \cdot \left(\frac{N^2}{2^j}\right)^{2k} \frac{(n-2k)!}{n!} \cdot \frac{1}{(k!)^2}.$$

The version of Stirling's formula due to Robbins [25] states that for all $s \geq 1$, $s! = \sqrt{2\pi s}(s/e)^s e^{r/s}$, where $1/(12s+1) < r/s < 1/12s$, and e is the base of the natural logarithm. We can assume that $n > 2k$. Thus,

$$\mathbf{P}\{A_k^j\} < (2m)^k \cdot (j+1)2^{j-1} \cdot \left(\frac{N^2}{2^j}\right)^{2k} \sqrt{\frac{n-2k}{n}} \left(\frac{n-2k}{e}\right)^{n-2k} \left(\frac{e}{n}\right)^n \frac{e^{2k}}{k^{2k+1}} \frac{r}{2\pi},$$

where the error term $r = e^{1/12(n-2k)} e^{-1/(12n+1)} e^{-2/(12k+1)} < e^{1/12}$. Hence $r/2\pi < 1/4$. Since $k \leq N^2/2^j$, $2^j \leq N^2/k \leq 16m/k$, and $2^{j-1} \leq 8m/k$. Since $n-2k < n$,

$$\begin{aligned} \mathbf{P}\{A_k^j\} &< (2m)^{k+1} (j+1) \left(\frac{1}{2^j}\right)^{2k} e^{4k} \left(\frac{1}{k}\right)^{2(k+1)} \\ &< \left(\frac{(j+1)\sqrt{2m}}{k} \left(\frac{e^2}{2^j}\right)^{\frac{k}{k+1}}\right)^{2(k+1)}. \end{aligned}$$

If $j=2$ then $e^2 > 2^j$, and hence $(e^2/2^j)^{k/(k+1)} \leq e^2/2^j \leq e^2/2\sqrt{2^j}$. If $j \geq 3$ then $e^2 < 2^j$, and since $k \geq 1$, $(e^2/2^j)^{k/(k+1)} \leq \sqrt{e^2/2^j} < e^2/2\sqrt{2^j}$. Therefore,

$$\mathbf{P}\{A_k^j\} < \left(\frac{e^2(j+1)}{\sqrt{2k}} \sqrt{\frac{m}{2^j}}\right)^{2(k+1)}. \quad (1)$$

Define $k_j = e^2(j + 1)\sqrt{m/2^j}$. Substituting k_j into (1), and since $m \geq n/4 \geq N^2/16$ and $j \geq 2$,

$$\mathbf{P}\{A_{k_j}^j\} < \left(\frac{1}{\sqrt{2}}\right)^{2(1+e^2(j+1)\sqrt{m/2^j})} \leq \left(\frac{1}{2}\right)^{1+(3/4)e^2\sqrt{N^2/2^j}}$$

The probability that the event $A_{k_j}^j$ occurs for some j , $2 \leq j \leq 2 \log N$,

$$\mathbf{P}\left\{\bigcup_{j=2}^{2 \log N} A_{k_j}^j\right\} < \frac{1}{2} \sum_{j=2}^{2 \log N} \left(\frac{1}{2}\right)^{(3/4)e^2\sqrt{N^2/2^j}} \tag{2}$$

By induction on X , the following can be proved.

$$\forall b \geq \frac{1}{\sqrt{2}-1}, \quad \sum_{j=2}^X \left(\frac{1}{2}\right)^{b\sqrt{2^{X-j}}} < \left(\frac{1}{2}\right)^{b-1}$$

Applying this fact to (2) with $X = 2 \log N$ and $b = \frac{3}{4}e^2 (> 1/(\sqrt{2}-1))$, and since $N^2 = 2^X$,

$$\mathbf{P}\left\{\bigcup_{j=2}^{2 \log N} A_{k_j}^j\right\} < \left(\frac{1}{2}\right)^{(3/4)e^2} \tag{3}$$

Thus,

$$\mathbf{P}\left\{\bigcap_{j=2}^{2 \log N} \overline{A_{k_j}^j}\right\} = 1 - \mathbf{P}\left\{\bigcup_{j=2}^{2 \log N} A_{k_j}^j\right\} > 1 - \left(\frac{1}{2}\right)^{(3/4)e^2} > 0.9785.$$

This says that for the random grid layout π , with (high) positive probability, $A_{k_j}^j$ does not occur for all j , $2 \leq j \leq 2 \log N$. Thus, there exists a grid layout π' of G such that $A_{k_j}^j$ does not occur for all j . That is, in each (a, b) -subgraph there is no completely crossing matching with at least k_{a+b} edges (with respect to π'). Now, apply Lemma 1 for each row or column subgraph, and Lemma 4 for each positive or negative subgraph. We obtain a layer assignment for the edges in each (a, b) -subgraph with at most k_{a+b} layers.

For each pair a, b ($1 \leq a, b \leq \log N$), edges in different positive (a, b) -subgraphs cannot intersect, and thus can share the same set of layers. Similarly for negative (a, b) -subgraphs. For each j , there are at most $j - 1$ pairs (a, b) with $a + b = j$ and $1 \leq a, b \leq \log N$. Thus the j -level positive and negative subgraphs take up at most $2(j - 1) \cdot k_j$ layers. Similarly the j -level row subgraphs can share the same set of layers, as can the j -level column subgraphs. Thus the j -level subgraphs take up at most $2j \cdot k_j$ layers, and the total number of layers is at most

$$\sum_{j=2}^{2 \log N} 2j \cdot k_j = 2e^2\sqrt{m} \sum_{j=2}^{2 \log N} \frac{j(j+1)}{\sqrt{2^j}} \leq 2(19\sqrt{2} + 28)e^2\sqrt{m} < 811\sqrt{m}. \quad \square$$

Note that in the proof of Theorem 5, little effort is made to reduce the constant in the $\mathcal{O}(\sqrt{m})$ bound. With a more judicious choice of k_j , for example, the constant can be improved. Consider the following Las Vegas algorithm to compute the drawing whose existence is proved in Theorem 5.

Algorithm. COMPUTE DRAWING (*input*: graph $G = (V, E)$)

0. Let $n = |V|$, $m = |E|$, and add $\lceil \sqrt{n} \rceil^2 - n$ isolated vertices to G .

repeat at most $\log n$ times:

1. Let π be a random $\lceil \sqrt{n} \rceil \times \lceil \sqrt{n} \rceil$ grid layout of G .
2. Determine the assignment of edges to layers as described in Theorem 5.
3. If the total number of layers is at most $811\sqrt{m}$ then halt.

Corollary 6. *The algorithm COMPUTE DRAWING, given a graph $G = (V, E)$ with $n = |V|$ vertices and $m = |E|$ edges, will in $\mathcal{O}(m \log^3 n \log \log n)$ time (with high probability) position the vertices of G in a $\lceil \sqrt{n} \rceil \times \lceil \sqrt{n} \rceil$ grid, and partition the edges of G into $\mathcal{O}(\sqrt{m})$ layers, such that no two edges in a single layer cross, and with each edge drawn with at most one bend.*

Proof. For each iteration of the above algorithm, we say the algorithm *fails* if the randomly chosen grid layout π does not admit a drawing with at most $811\sqrt{m}$ layers. By (3), the probability of failure is at most $2^{-3e^2/4}$. Thus the probability of failure $\log n$ times is at most $2^{(-3e^2/4)\log n} = n^{-3e^2/4}$, which tends to 0 as $n \rightarrow \infty$. Thus, with probability tending to 1 as $n \rightarrow \infty$, the above algorithm will determine the required drawing.

Consider the time taken for Step 2 in one iteration. For each pair a, b ($1 \leq a, b \leq \log N$), it is easily seen that Lemmas 1 or 4 can be simultaneously applied for all positive (a, b) -subgraphs, and similarly for all negative, row, and column (a, b) -subgraphs. Thus the time for one iteration is $\sum_{a,b} \mathcal{O}(m_{a,b} \log \log n)$, where $m_{a,b}$ is the number of edges in (a, b) -regional subgraphs, which is $\mathcal{O}(m \log^2 n \log \log n)$. Thus the time for $\log n$ iterations is $\mathcal{O}(m \log^3 n \log \log n)$. \square

As a final note we mention two important differences between the proof by Malitz [22] that $\text{bt}(G) \in \mathcal{O}(\sqrt{m})$ and our proof of Theorem 5. First, we do not assume that $j \leq k$, as is the case in [22, p. 76] (also see [21, p. 92]). Furthermore, we do not employ an explicit drawing of a complete graph $K_{\sqrt{n}}$ for large values of j , as is the case in [22]. See [28] for a related method.

Note added in proof

Subsequent to this research, Eppstein [14] has shown that (graph-theoretic) thickness and geometric thickness are asymptotically unrelated. In particular, for every t , there exists a graph with thickness three and geometric thickness at least t .

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