

# Vertex Partitions of Chordal Graphs

David R. Wood

DEPARTAMENT DE MATEMÀTICA APLICADA II  
UNIVERSITAT POLITÈCNICA DE CATALUNYA  
BARCELONA, SPAIN  
E-mail: david.wood@upc.edu

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**Abstract:** A  $k$ -tree is a chordal graph with no  $(k + 2)$ -clique. An  $\ell$ -tree-partition of a graph  $G$  is a vertex partition of  $G$  into 'bags,' such that contracting each bag to a single vertex gives an  $\ell$ -tree (after deleting loops and replacing parallel edges by a single edge). We prove that for all  $k \geq \ell \geq 0$ , every  $k$ -tree has an  $\ell$ -tree-partition in which each bag induces a connected  $\lfloor k/(\ell + 1) \rfloor$ -tree. An analogous result is proved for oriented  $k$ -trees. © 2006 Wiley Periodicals, Inc. *J Graph Theory* 53: 167–172, 2006

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## 1. INTRODUCTION

Let  $G$  be an (undirected, simple, finite) graph with vertex set  $V(G)$  and edge set  $E(G)$ . Let  $\Delta(G)$  be the maximum degree of  $G$ . The neighborhood of a vertex  $v$  of  $G$  is denoted by  $N(v) = \{w \in V(G) : vw \in E(G)\}$ . A *chord* of a cycle  $C$  is an edge

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not in  $C$  whose endpoints are both in  $C$ .  $G$  is *chordal* if every cycle on at least four vertices has a chord. A  $k$ -clique ( $k \geq 0$ ) is a set of  $k$  pairwise adjacent vertices. A  $k$ -tree is a chordal graph with no  $(k + 2)$ -clique. The *tree-width* of  $G$ , denoted by  $\text{tw}(G)$ , is the minimum  $k$  such that  $G$  is a subgraph of a  $k$ -tree. It is well known that this definition agrees with the standard definition of tree-width in terms of tree decompositions. It is also well known that  $G$  is a  $k$ -tree<sup>1</sup> if and only if  $V(G) = \emptyset$ , or  $G$  has a vertex  $v$  such that  $G \setminus v$  is a  $k$ -tree, and  $N(v)$  is a  $k'$ -clique for some  $k' \leq k$ .

Let  $G$  and  $H$  be graphs. The elements of  $V(H)$  are called *nodes*. Let  $\{H_x \subseteq V(G) : x \in V(H)\}$  be a set of subsets of  $V(G)$  indexed by the nodes of  $H$ . Each set  $H_x$  is called a *bag*. The pair  $(H, \{H_x \subseteq V(G) : x \in V(H)\})$  is an  $H$ -partition of  $G$  if:

- for every vertex  $v$  of  $G$ , there is a node  $x$  of  $H$  with  $v \in H_x$ , and
- for all distinct nodes  $x$  and  $y$  of  $H$ ,  $H_x \cap H_y = \emptyset$ , and
- for every edge  $vw$  of  $G$ , either
  - there is a node  $x$  of  $H$  with  $v \in H_x$  and  $w \in H_x$ , or
  - there is an edge  $xy$  of  $H$  with  $v \in H_x$  and  $w \in H_y$ .

For brevity we say  $H$  is a partition of  $G$ . A  $k$ -tree-partition is an  $H$ -partition for some  $k$ -tree  $H$ . A *tree-partition* is a 1-tree-partition. Tree-partitions were independently introduced by Seese [16] and Halin [12], and have since been investigated by a number of authors [2,3,6,7,11,12,16]. The main property of tree-partitions that has been studied is the maximum cardinality of a bag, called the *width* of the tree-partition. The minimum width over all tree-partitions of a graph  $G$  is the *tree-partition-width*<sup>2</sup> of  $G$ , denoted by  $\text{tpw}(G)$ .

A graph with bounded degree has bounded tree-partition-width if and only if it has bounded tree-width [7]. In particular, Seese [16] proved the lower bound,

$$2 \cdot \text{tpw}(G) \geq \text{tw}(G) + 1,$$

which is tight for even complete graphs. The best known upper bound is

$$\text{tpw}(G) \leq 2(\text{tw}(G) + 1)(9 \Delta(G) - 1),$$

which was obtained by the author [18] using a minor improvement to a similar result by an anonymous referee of the paper by Ding and Oporowski [6]. See [1,5,8,9,14] for other results related to tree-width and vertex partitions<sup>3</sup>.

<sup>1</sup> In the most common definition of  $k$ -tree,  $N(v)$  is required to be a  $k$ -clique. Working in the slightly larger class of graphs enables cleaner results.

<sup>2</sup> Tree-partition-width has also been called *strong tree-width* [3,16].

<sup>3</sup> Vertex partitions also provide an approach for attacking Hadwiger's conjecture, which states that every graph  $G$  with no  $K_{t+1}$  minor has chromatic number  $\chi(G) \leq t$ . No  $\chi(G) \leq \mathcal{O}(t)$  bound is currently known. Reed and Seymour [15] observed that "perhaps it is true" that  $G$  has an  $H$ -partition, such that  $H$  is chordal and each bag induces a connected bipartite subgraph of  $G$ . This would imply that  $\chi(H) = \omega(H) \leq t$ , and thus  $\chi(G) \leq 2t$ . Note that we only need  $H$  to be perfect for this conclusion to be reached.

Tree-partition-width is not bounded above by any function solely of tree-width. For example, wheel graphs have bounded tree-width and unbounded tree-partition-width, as observed by Bodlaender and Engelfriet [3]. Thus, it seems unavoidable that the maximum degree appears in an upper bound on the tree-partition-width. This fact, along with other applications, motivated Dujmović et al. [10] to study the structure of the bags in a tree-partition. In this article we continue this approach, and prove the following result (in Section 2).

**Theorem 1.** *Let  $k$  and  $\ell$  be integers with  $k \geq \ell \geq 0$ . Let  $t = \lfloor k/(\ell + 1) \rfloor$ . Every  $k$ -tree  $G$  has an  $\ell$ -tree-partition in which each bag induces a connected  $t$ -tree.*

It is easily seen that Theorem 1 is tight for  $G = K_{k+1}$  and for all  $\ell$ . Note that Theorem 1 can be interpreted as a statement about chromomorphisms [13].

Dujmović et al. [10] proved that every  $k$ -tree has a tree-partition in which each bag induces a  $(k - 1)$ -tree. Thus Theorem 1 with  $\ell = 1$  improves this result. That said, the tree-partition of Dujmović et al. [10] has a number of additional properties that were important for the intended application. We generalize these additional properties in Section 3. The price paid is that each bag may now induce a  $(k - \ell)$ -tree, thus matching the result of Dujmović et al. [10] for  $\ell = 1$ . Note that the proof of Dujmović et al. [10] uses a different construction to the one given here.

## 2. PROOF OF THEOREM 1

We proceed by induction on  $|V(G)|$ . If  $V(G) = \emptyset$ , then the result holds with  $V(H) = \emptyset$  regardless of  $k$  and  $\ell$ . Now suppose that  $|V(G)| \geq 1$ . Thus  $G$  has a vertex  $v$  such that  $G \setminus v$  is a  $k$ -tree, and  $N(v)$  is a  $k'$ -clique for some  $k' \leq k$ . By induction,  $G \setminus v$  has an  $\ell$ -tree-partition  $H$  in which each bag induces a connected  $t$ -tree. Let  $C = \{x \in V(H) : N(v) \cap H_x \neq \emptyset\}$ . Since  $N(v)$  is a clique,  $C$  is a clique of  $H$  (by the definition of  $H$ -partition). Since  $H$  is an  $\ell$ -tree,  $|C| \leq \ell + 1$ .

**Case 1.**  $|C| \leq \ell$ : Add one new node  $y$  to  $H$  adjacent to each node  $x \in C$ . Since  $C$  is a clique of  $H$  and  $|C| \leq \ell$ ,  $H$  remains an  $\ell$ -tree. Let  $H_y = \{v\}$ . The other bags remain unchanged. Since  $t \geq 0$ ,  $H_y$  induces a connected  $t$ -tree ( $= K_1$ ) in  $G$ . Thus  $H$  is now a partition of  $G$  in which each bag induces a connected  $t$ -tree in  $G$ .

**Case 2.**  $|C| = \ell + 1$ : There is a node  $y \in C$  such that  $|N(v) \cap H_y| \leq t$ , as otherwise  $|N(v)| \geq (t + 1)|C| = (\lfloor k/(\ell + 1) \rfloor + 1)(\ell + 1) \geq k + 1$ . Add  $v$  to the bag  $H_y$ . Let  $u \in N(v) \cap H_y$ . Every neighbor of  $v$  not in  $H_y$  is adjacent to  $u$  (in  $G \setminus v$ ). Thus  $H$  is a partition of  $G$ .  $H_y$  induces a connected  $t$ -tree in  $G$ , since  $H_y \setminus \{v\}$  induces a connected  $t$ -tree in  $G \setminus v$ , and the neighborhood of  $v$  in  $H_y$  is a clique of at least one and at most  $t$  vertices. The other bags do not change. Thus each bag of  $H$  induces a connected  $t$ -tree in  $G$ . ■

### 3. ORIENTED PARTITIONS

Let  $G$  be an oriented graph with arc set  $A(G)$ . Let  $\widehat{G}$  be the underlying undirected graph of  $G$ . The in- and out-neighborhoods of a vertex  $v$  of  $G$  are respectively denoted by  $N^-(v) = \{u \in V(G) : uv \in A(G)\}$  and  $N^+(v) = \{w \in V(G) : vw \in A(G)\}$ . It is easily seen that an (undirected) graph  $G$  is a  $k$ -tree if and only if there is an acyclic orientation of  $G$  such that for every vertex  $v$  of  $G$ ,  $N^-(v)$  is a  $k'$ -clique for some  $k' \leq k$ . An oriented graph with this property is called an *oriented  $k$ -tree*. Let  $G$  and  $H$  be oriented graphs. An *oriented  $H$ -partition* of  $G$  is an  $\widehat{H}$ -partition of  $\widehat{G}$  such that for every arc  $xy$  of  $H$ , and for every edge  $vw$  of  $\widehat{G}$  with  $v \in H_x$  and  $w \in H_y$ ,  $vw$  is oriented from  $v$  to  $w$ . This concept is similar to an oriented homomorphism; see [4,17] for example.

**Theorem 2.** *Let  $k$  and  $\ell$  be integers with  $k \geq \ell \geq 0$ . Let  $t = k - \ell$ . Every oriented  $k$ -tree  $G$  has an oriented  $\ell$ -tree partition  $H$  in which each bag induces a weakly connected oriented  $t$ -tree in  $G$ . Moreover, for each node  $x$  of  $H$ , the set of vertices  $Q(x) = \bigcup\{N^-(v) \setminus H_x : v \in H_x\}$  is a  $k'$ -clique of  $G$  for some  $k' \leq k$ .*

The construction in the proof of Theorem 2 differs from that of Theorem 1 in only the choice of the node  $y$  in Case 2.

**Proof.** We proceed by induction on  $|V(G)|$ . If  $V(G) = \emptyset$ , then the result holds with  $V(H) = \emptyset$  regardless of  $k$  and  $\ell$ . Now suppose that  $|V(G)| \geq 1$ . Since  $G$  is acyclic, there is a vertex  $v$  of  $G$  such that  $N^+(v) = \emptyset$ ,  $N^-(v)$  is a  $k'$ -clique for some  $k' \leq k$ , and  $G \setminus v$  is an oriented  $k$ -tree. By induction, there is an oriented  $\ell$ -tree-partition  $H$  of  $G \setminus v$  in which each bag induces a weakly connected oriented  $t$ -tree in  $G \setminus v$ . Moreover, for every node  $x$  of  $H$ ,  $Q(x)$  is a  $k'$ -clique for some  $k' \leq k$ . Let  $C = \{x \in V(H) : N^-(v) \cap H_x \neq \emptyset\}$ . Since  $N^-(v)$  is a clique,  $C$  is a clique of  $H$ . Since  $H$  is an oriented  $\ell$ -tree,  $|C| \leq \ell + 1$ .

**Case 1.**  $|C| \leq \ell$ : Add one new node  $y$  to  $H$  adjacent to each node  $x \in C$ . Orient each new edge from  $x$  to  $y$ . Obviously  $H$  remains acyclic. Since  $C$  is a clique of  $H$  and  $|C| \leq \ell$ ,  $H$  remains an oriented  $\ell$ -tree. Let  $H_y = \{y\}$ . The other bags are unchanged. Since  $t \geq 0$ ,  $H_y$  induces a weakly connected oriented  $t$ -tree ( $=K_1$ ) in  $G$ . All edges of  $G$  that are incident to a vertex in  $H_y$  are oriented into the vertex in  $H_y$ . Thus  $H$  is now an oriented partition of  $G$  in which each bag induces a weakly connected oriented  $t$ -tree in  $G$ . Now  $Q(y) = N^-(v)$ , which is a  $k'$ -clique for some  $k' \leq k$ .  $Q(x)$  is unchanged for nodes  $x \neq y$ . Hence the theorem is satisfied.

**Case 2.**  $|C| = \ell + 1$ : The clique  $C$  induces an acyclic tournament in  $H$ . Let  $y$  be the sink of this tournament. Since  $|N^-(v) \cap H_x| \geq 1$  for every node  $x \in C \setminus \{y\}$ ,  $|N^-(v) \cap H_y| \leq k' - (|C| - 1) \leq k - \ell = t$ . Add  $v$  to the bag  $H_y$ .

Consider a neighbor  $u$  of  $v$ . Since  $N^+(v) = \emptyset$ ,  $uv$  is oriented from  $u$  to  $v$ . Say  $u \in H_z$  with  $z \neq y$ . Then  $z$  is in the clique  $C$ . Thus  $zy$  is an edge of  $H$ . Since  $y$  is a sink of  $C$ ,  $zy$  is oriented from  $z$  to  $y$ . Thus  $H$  is now an oriented partition of  $G$ .  $H_y$

induces a weakly connected oriented  $t$ -tree in  $G$ , since  $H_y \setminus \{v\}$  induces an oriented  $t$ -tree in  $G \setminus v$ , and the in-neighborhood of  $v$  in  $H_y$  is a clique of at least one and at most  $t$  vertices. The other bags do not change. Thus each bag of  $H$  induces a weakly connected oriented  $t$ -tree in  $G$ .

$Q(y)$  is not changed by the addition of  $v$  to  $H_y$ , as there is at least one vertex  $u \in N^-(v) \cap H_y$ , and any vertex in  $N^-(v) \setminus H_y$  is also in  $N^-(u) \setminus H_y$ . For nodes  $x \neq y$ ,  $Q(x)$  is unchanged by the addition of  $v$  to  $H_y$ , since  $v$  is not in the in-neighborhood of any vertex. Hence the theorem is satisfied. ■

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