

Queue Layouts of Graph Products and Powers[†]

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A *k*-queue layout of a graph G consists of a linear order σ of $V(G)$, and a partition of $E(G)$ into k sets, each of which contains no two edges that are nested in σ . This paper studies queue layouts of graph products and powers.

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1 Introduction

Let G be a graph. (All graphs considered are finite, simple and undirected.) The vertex and edge sets of G are denoted by $V(G)$ and $E(G)$, respectively. The minimum and maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The *density* of G is $\eta(G) := |E(G)|/|V(G)|$.

A *vertex ordering* of G is a bijection $\sigma : V(G) \rightarrow \{1, 2, \dots, |V(G)|\}$. In a vertex ordering σ of G , let $L_\sigma(e)$ and $R_\sigma(e)$ denote the endpoints of each edge $e \in E(G)$ such that $\sigma(L_\sigma(e)) < \sigma(R_\sigma(e))$. Where the vertex ordering σ is clear from the context, we will abbreviate $L_\sigma(e)$ and $R_\sigma(e)$ by L_e and R_e , respectively. For edges e and f of G with no endpoint in common, there are the following three possible relations with respect to σ , as illustrated in Figure 1:

- (a) e and f *nest* if $\sigma(L_e) < \sigma(L_f) < \sigma(R_f) < \sigma(R_e)$,
- (b) e and f *cross* if $\sigma(L_e) < \sigma(L_f) < \sigma(R_e) < \sigma(R_f)$,
- (c) e and f are *disjoint* if $\sigma(L_e) < \sigma(R_e) < \sigma(L_f) < \sigma(R_f)$.

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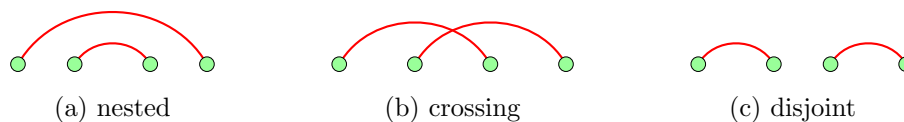


Fig. 1: Relationships between pairs of edges with no common endpoint in a vertex ordering.

A *queue* in σ is a set of edges $Q \subseteq E(G)$ such that no two edges in Q are nested. Observe that when traversing σ from left to right, the left and right endpoints of the edges in a queue are reached in first-in-first-out order—hence the name ‘queue’. Observe that $Q \subseteq E(G)$ is a queue if and only if for all edges $e, f \in Q$,

$$\begin{aligned} \sigma(L_e) \leq \sigma(L_f) \text{ and } \sigma(R_e) \leq \sigma(R_f) , \\ \text{or } \sigma(L_f) \leq \sigma(L_e) \text{ and } \sigma(R_f) \leq \sigma(R_e) . \end{aligned} \quad (1)$$

A k -*queue layout* of G is a pair

$$(\sigma, \{Q_1, Q_2, \dots, Q_k\})$$

where σ is a vertex ordering of G , and $\{Q_1, Q_2, \dots, Q_k\}$ is a partition of $E(G)$, such that each Q_i is a queue in σ . The *queue-number* of a graph G , denoted by $\text{qn}(G)$, is the minimum k such that there is a k -queue layout of G .

Queue layouts were introduced by Heath et al. [15, 19]. Applications of queue layouts include sorting permutations [12, 20, 22, 24, 27], parallel process scheduling [3], matrix computations [23], and graph drawing [4, 6]. Other aspects of queue layouts have been studied in the literature [7, 9, 10, 13, 25, 26]. Queue layouts of directed graphs [5, 11, 17, 18] and posets [16] have also been investigated.

Table 1 describes the best known upper bounds on the queue-number of various classes of graphs. Planar graphs are an interesting class of graphs for which it is not known whether the queue-number is bounded (see [6, 23]).

This paper studies queue layouts of graph products and graph powers. To prove optimality we use the following lower bound by Heath and Rosenberg [19]. See Pemmaraju [23] and Dujmović and Wood [9] for slightly more exact lower bounds.

Lemma 1 ([19]) *Every graph G has queue-number $\text{qn}(G) > \eta(G)/2$.*

This paper is organised as follows. In Section 2 we introduce the concepts of strict queue layout and strict queue-number. Many of the upper bounds on the queue-number that are presented in later sections will be expressed as functions of the strict queue-number. In Section 3 we prove bounds on the queue-number of the power of a graph in terms of the queue-number of the underlying graph. In Section 4 we define the graph products that will be studied in later sections. In Section 5 we study the queue-number of the cartesian product of graphs. Finally in Section 6 we study the queue-number of the direct and strong products of graphs.

[‡] Dujmović and Wood [8] gave a simple proof of this result.

Tab. 1: Upper bounds on the queue-number.

graph family	queue-number	reference
n vertices	$\lfloor \frac{n}{2} \rfloor$	Heath and Rosenberg [19]
m edges	$e\sqrt{m}$	Dujmović and Wood [9]
tree-width w	$3^w \cdot 6^{(4^w - 3w - 1)/9} - 1$	Dujmović <i>et al.</i> [6]
tree-width w , max. degree Δ	$36\Delta w$	Wood [29]
path-width p	p	Dujmović <i>et al.</i> [6]
band-width b	$\lfloor \frac{b}{2} \rfloor$	Heath and Rosenberg [19]
track-number t	$t - 1$	Dujmović <i>et al.</i> [6]
2-trees	3	Rengarajan and Veni Madhavan [25] [‡]
k -ary butterfly	$\lfloor \frac{k}{2} \rfloor + 1$	Hasunuma [14]
d -ary de Bruijn	d	Hasunuma [14]
Halin	3	Ganley [13]
X-trees	2	Heath and Rosenberg [19]
outerplanar	2	Heath <i>et al.</i> [15]
arched levelled planar	1	Heath <i>et al.</i> [15]
trees	1	Heath and Rosenberg [19]

2 Strict Queue Layouts

Let σ be a vertex ordering of a graph G . We say an edge e is *inside* a distinct edge f , and e and f *overlap*, if

$$\sigma(L_f) \leq \sigma(L_e) < \sigma(R_e) \leq \sigma(R_f) .$$

A set of edges $Q \subseteq E(G)$ is a *strict queue* in σ if no edge in Q is inside another edge in Q . Alternatively, Q is a *strict queue* in σ if

$$\begin{aligned} &\sigma(L_e) < \sigma(L_f) \text{ and } \sigma(R_e) < \sigma(R_f) , \\ \text{or } &\sigma(L_f) < \sigma(L_e) \text{ and } \sigma(R_f) < \sigma(R_e) . \end{aligned} \tag{2}$$

Note that Equation (2) is obtained from Equation (1) by replacing “ \leq ” by “ $<$ ”.

Hence a strict queue is a set of edges, no two of which are nested or overlapping, as illustrated in Figure 2. Note that edges forming a ‘butterfly’ can be in a single strict queue.

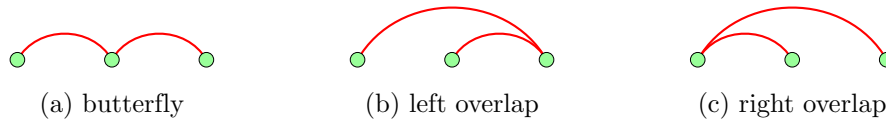


Fig. 2: Relationships between pairs of edges with a common endpoint in a vertex ordering.

A *strict k -queue layout* of G is a pair $(\sigma, \{Q_1, Q_2, \dots, Q_k\})$ where σ is a vertex ordering of G , and $\{Q_1, Q_2, \dots, Q_k\}$ is a partition of $E(G)$, such that each Q_i is a strict queue in σ . We sometimes write

$\text{queue}(e) = i$ for each edge $e \in Q_i$. The *strict-queue-number* of a graph G , denoted by $\text{sqn}(G)$, is the minimum k such that there is a strict k -queue layout of G .

Heath and Rosenberg [19] proved that a fixed vertex ordering of a graph G admits a k -queue layout of G if and only if it has no $(k + 1)$ -edge rainbow, where a *rainbow* is a set of pairwise nested edges, as illustrated in Figure 3(a). Consider the analogous problem for strict queues: assign the edges of a graph G to the minimum number of strict queues given a fixed vertex ordering σ of G . As illustrated in Figure 3(b), a *weak rainbow* in σ is a set of edges R such that for every pair of edges $e, f \in R$, e is inside f or f is inside e .

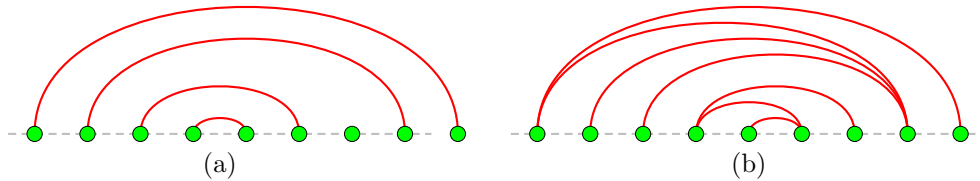


Fig. 3: (a) rainbow, (b) weak rainbow

Lemma 2 A vertex ordering of a graph G admits a strict k -queue layout of G if and only if it has no $(k + 1)$ -edge weak rainbow.

Proof: A strict k -queue layout has no $(k + 1)$ -edge weak rainbow since each edge of a weak rainbow must be in a distinct strict queue. Conversely, suppose we have a vertex ordering with no $(k + 1)$ -edge weak rainbow. For every edge $e \in E(G)$, let $\text{queue}(e)$ be one plus the maximum number of edges in a weak rainbow consisting of edges that are inside e . If e is inside f then $\text{queue}(e) < \text{queue}(f)$. Hence we have a valid strict queue assignment. The number of strict queues is at most k . \square

A *linear forest* is a graph in which every component is a path. The *linear arboricity* of a graph G , denoted by $\text{la}(G)$, is the minimum integer k such that $E(G)$ can be partitioned in k linear forests; see [1, 2, 30, 31]. We have the following lower bounds on $\text{sqn}(G)$.

Lemma 3 The strict queue-number of every graph G satisfies:

- (a) $\text{sqn}(G) \geq \text{la}(G) > \eta(G)$,
- (b) $\text{sqn}(G) \geq \text{la}(G) \geq \Delta(G)/2$, and
- (c) $\text{sqn}(G) \geq \delta(G)$.

Proof: Say Q is a strict queue in a vertex ordering σ of G . Every 2-edge path (u, v, w) in Q has $\sigma(u) < \sigma(v) < \sigma(w)$ (or $\sigma(w) < \sigma(v) < \sigma(u)$). Thus no vertex is incident to three edges in Q , and Q induces a linear forest. Hence $\text{la}(G) \leq \text{sqn}(G)$.

Since a linear forest in G has at most $|V(G)| - 1$ edges, $\text{la}(G) \geq |E(G)| / (|V(G)| - 1) > \eta(G)$. This proves (a). At most two edges incident to each vertex are a linear forest. Thus $\text{la}(G) \geq \Delta(G)/2$. This proves (b).

In every vertex ordering of G , every edge incident to the first vertex is in a distinct strict queue. Hence $\text{sqn}(G) \geq \delta(G)$. This proves (c). \square

Obviously a proper edge $(\Delta(G) + 1)$ -colouring [28] can be combined with a $\text{qn}(G)$ -queue layout to obtain a strict queue layout.

Lemma 4 Every graph G has strict queue-number $\text{sqn}(G) \leq (\Delta(G) + 1) \cdot \text{qn}(G)$. \square

3 Graph Powers

Let G be a graph, and let $d \in \mathbb{Z}^+$. The d -th power of G , denoted by G^d , is the graph with vertex set $V(G^d) = V(G)$, where $vw \in E(G^d)$ if and only if the distance between v and w in G is at most d . The following general result is similar to a theorem of Dujmović and Wood [10].

Theorem 1 For every graph G and $d \in \mathbb{Z}^+$,

$$\text{qn}(G^d) \leq \frac{(2 \text{sqn}(G))^{d+1} - 1}{2 \text{sqn}(G) - 1} - \text{sqn}(G) - 1 .$$

Proof: Let σ be the vertex ordering in a strict $\text{sqn}(G)$ -queue layout of G . Consider σ to be a vertex ordering of G^d . For every pair of vertices $v, w \in V(G)$ with $\sigma(v) < \sigma(w)$ and at distance $\ell \leq d$, fix a path $P(vw)$ from v to w in G with exactly ℓ edges. Suppose $P(vw) = (x_0, x_1, \dots, x_\ell)$, where $v = x_0$ and $w = x_\ell$. For each $1 \leq i \leq \ell$, let $\text{dir}(x_{i-1}x_i)$ be ‘+’ if $\sigma(x_{i-1}) < \sigma(x_i)$, and ‘-’ otherwise. Let $f(vw)$ be the vector

$$f(vw) = \left[(\text{queue}(x_{i-1}x_i), \text{dir}(x_{i-1}x_i)) : 1 \leq i \leq \ell \right] .$$

Consider two edges $vw, pq \in E(G^d)$ with $f(vw) = f(pq)$. Then $|P(vw)| = |P(pq)|$. Let $P(vw) = (x_0, x_1, \dots, x_\ell)$ and $P(pq) = (y_0, y_1, \dots, y_\ell)$. We have $\text{dir}(x_0x_1) = \text{dir}(y_0y_1)$ and $\text{queue}(x_0x_1) = \text{queue}(y_0y_1)$. Thus $x_0 \neq y_0$. Without loss of generality $\sigma(x_0) < \sigma(y_0)$. By Equation (2), $\sigma(x_1) < \sigma(y_1)$. In general, $\sigma(x_{i-1}) < \sigma(y_{i-1})$ implies $\sigma(x_i) < \sigma(y_i)$, since $\text{queue}(x_{i-1}x_i) = \text{queue}(y_{i-1}y_i)$ and $\text{dir}(x_{i-1}x_i) = \text{dir}(y_{i-1}y_i)$. By induction, $\sigma(x_i) < \sigma(y_i)$ for all $0 \leq i \leq \ell$. In particular, $\sigma(w) < \sigma(q)$. Thus vw and pq can be in the same strict queue. If we partition the edges of G^d by the value of f we obtain a strict queue layout of G^d . The number of queues is

$$\sum_{\ell=1}^d (2 \text{sqn}(G))^\ell = \frac{(2 \text{sqn}(G))^{d+1} - 1}{2 \text{sqn}(G) - 1} - 1 .$$

Observe that for the edges of G we have counted $2 \text{sqn}(G)$ queues. Of course we need only $\text{sqn}(G)$ queues. Thus the total number of queues is as claimed. \square

3.1 Powers of Paths and Cycles

In a vertex ordering σ of a graph G , the *width* of an edge e is $\sigma(R_e) - \sigma(L_e)$. The *bandwidth* of σ is the maximum width of an edge of G . The *bandwidth* of G , denoted by $\text{bw}(G)$, is the minimum bandwidth of a vertex ordering of G . Alternatively, $\text{bw}(G) = \min\{k : G \subseteq P_n^k\}$ for every n -vertex graph G .

Heath and Rosenberg [19] observed that edges whose widths differ by at most one are not nested. Thus $\text{qn}(G) \leq \lceil \text{bw}(G)/2 \rceil$, as mentioned in Table 1. In a vertex ordering, edges with the same width are not nested or overlapping, and thus form a strict queue. The next lemma follows.

Lemma 5 *Every graph G has strict queue-number $\text{sqn}(G) \leq \text{bw}(G)$.* \square

We have the following results that give more precise bounds on the queue-number and strict-queue-number of powers of paths and cycles than Theorem 1.

Lemma 6 *The k -th power of a path P_n ($n \geq k + 1$) has queue-number $\text{qn}(P_n^k) = \lceil k/2 \rceil$ and strict queue-number $\text{sqn}(P_n^k) = k$*

Proof: The bandwidth of a graph G can be thought of as the minimum integer k such that $G \subseteq P_n^k$. Thus the upper bound is nothing more than the result $\text{qn}(G) \leq \lceil \text{bw}(G)/2 \rceil$ of Heath and Rosenberg [19]. The lower bound follows since P_n^k contains a $(k + 1)$ -clique, which contains $\lceil k/2 \rceil$ pairwise nested edges in any vertex ordering, all of which must be assigned to distinct queues.

The natural vertex-ordering of P_n^k has no $(k + 1)$ -edge weak rainbow. Thus $\text{sqn}(P_n^k) \leq k$ by Lemma 2. The lower bound follows since P_n^k contains a $(k + 1)$ -clique, which contains a k -edge weak rainbow in any vertex ordering. \square

A graph is *unicyclic* if every connected component has at most one cycle. Heath and Rosenberg [19] proved that any unicyclic graph has a 1-queue layout. In particular, every cycle has a 1-queue layout. More generally,

Lemma 7 *The k -th power of a cycle C_n ($n \geq 2k$) has queue-number $\frac{k}{2} < \text{qn}(C_n^k) \leq k$, and strict queue-number $\text{sqn}(C_n^k) = 2k$.*

Proof: Observe that $\delta(C_n^k) = \Delta(C_n^k) = 2k$ and $\eta(C_n^k) = k$. Thus the claimed lower bounds follow from Lemmata 1 and 3. For the upper bounds, say $C_n = (v_1, v_2, \dots, v_n)$. By considering the vertex ordering

$$(v_1, v_n; v_2, v_{n-1}; \dots; v_i, v_{n-i+1}; \dots; v_{\lfloor n/2 \rfloor}, v_{\lceil n/2 \rceil}) , \quad (3)$$

we see that $C_n^k \subseteq P_n^{2k}$. The result follows from Lemma 6. \square

4 Graph Products

Let G_1 and G_2 be graphs. Below we define a number of graph products whose vertex set is

$$V(G_1) \times V(G_2) = \{(a, v) : a \in V(G_1), v \in V(G_2)\} .$$

We classify a potential edge $(a, v)(b, w)$ as follows:

- G_1 -edge: $ab \in E(G_1)$ and $v = w$.
- G_2 -edge: $a = b$ and $vw \in E(G_2)$.
- direct edge: $ab \in E(G_1)$ and $vw \in E(G_2)$.

The *cartesian product* $G_1 \square G_2$ consists of the G_1 -edges and the G_2 -edges. The *direct product* $G_1 \times G_2$ consists of the direct edges. The *strong product* $G_1 \boxtimes G_2$ consists of the G_1 -edges, the G_2 -edges, and the direct edges. That is, $G_1 \boxtimes G_2 = (G_1 \square G_2) \cup (G_1 \times G_2)$. Note that other names abound for these graph products. Our notation is taken from the survey by Klavžar [21]. Assuming isomorphic graphs are equal, each of the above three products are associative, and for instance, $G_1 \square G_2 \square \dots \square G_d$ is well-defined. Figure 4 illustrates these three types of graph products.

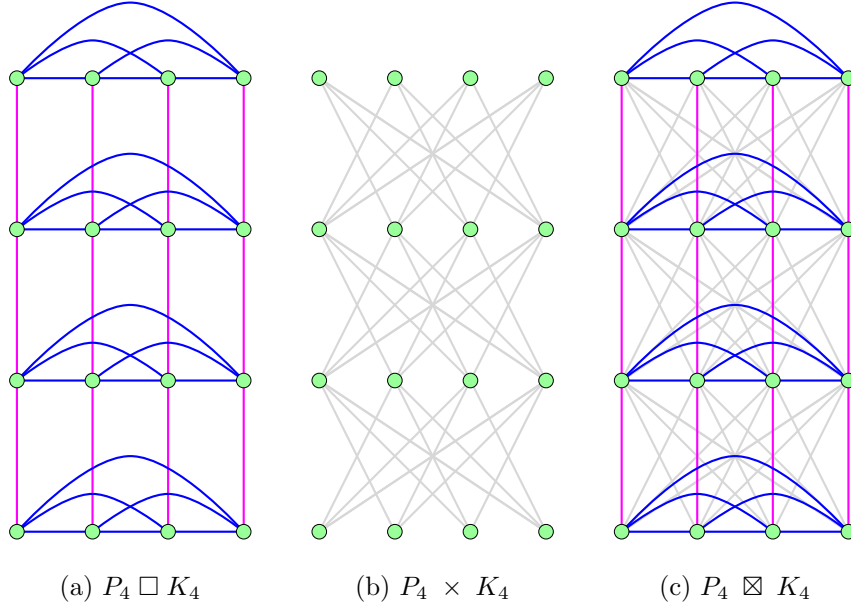


Fig. 4: Examples of graph products: (a) cartesian, (b) direct, (c) strong.

The following lemma is well-known and easily proved.

Lemma 8 For all graphs G_1 and G_2 , the density satisfies

(a) $\eta(G_1 \square G_2) = \eta(G_1) + \eta(G_2)$,

(b) $\eta(G_1 \times G_2) = 2\eta(G_1) \cdot \eta(G_2)$,

(c) $\eta(G_1 \boxtimes G_2) = 2\eta(G_1) \cdot \eta(G_2) + \eta(G_1) + \eta(G_2)$.

5 The Cartesian Product

We have the following bounds on the queue-number of a cartesian product. In a vertex ordering σ of a graph product, we abbreviate $\sigma((v, a))$ by $\sigma(v, a)$.

Theorem 2 For all graphs G and H ,

$$\text{qn}(G \square H) \leq \text{sqn}(G) + \text{qn}(H) .$$

Furthermore, if for some constant c we have $\text{sqn}(G) \leq c \cdot \eta(G)$ and $\text{qn}(H) \leq c \cdot \eta(H)$, then

$$\text{qn}(G \square H) \geq \frac{1}{2c} (\text{sqn}(G) + \text{qn}(H)) .$$

Proof: First we prove the upper bound. Let σ be the vertex ordering in a strict $\text{sqn}(G)$ -queue layout of G . Let π be the vertex ordering in a $\text{qn}(H)$ -queue layout of H . Let ϕ be the vertex ordering of $G \square H$ in which $\phi(v, a) < \phi(w, b)$ if and only if $\sigma(v) < \sigma(w)$, or $v = w$ and $\pi(a) < \pi(b)$.

For all edges e of G and for all vertices a of H , we have $\phi(L_e, a) < \phi(R_e, a)$. Similarly, for all edges e of H and for all vertices v of G , we have $\phi(v, L_e) < \phi(v, R_e)$.

Consider two G -edges $(L_e, a)(R_e, a)$ and $(L_f, b)(R_f, b)$ of $G \square H$, for which e and f are in the same strict queue of G . By Equation (2), without loss of generality, $\sigma(L_e) < \sigma(L_f)$ and $\sigma(R_e) < \sigma(R_f)$. Thus $\phi(L_e, a) < \phi(L_f, b)$ and $\phi(R_e, a) < \phi(R_f, b)$. Hence for each strict queue in G , the corresponding G -edges of $G \square H$ form a strict queue in ϕ .

Consider two H -edges $(v, L_e)(v, R_e)$ and $(w, L_f)(w, R_f)$ of $G \square H$, for which e and f are in the same queue of H . By Equation (1), without loss of generality, $\pi(L_e) \leq \pi(L_f)$ and $\pi(R_e) \leq \pi(R_f)$. First suppose that $\sigma(v) \leq \sigma(w)$. Then $\phi(v, L_e) \leq \phi(w, L_f)$ and $\phi(v, R_e) \leq \phi(w, R_f)$. Thus $(v, L_e)(v, R_e)$ and $(w, L_f)(w, R_f)$ are not nested in ϕ . Now suppose that $\sigma(w) < \sigma(v)$. Then $\phi(w, L_f) < \phi(w, R_f) < \phi(v, L_e) < \phi(v, R_e)$. Thus $(v, L_e)(v, R_e)$ and $(w, L_f)(w, R_f)$ are disjoint. Thus for each queue in H , the corresponding H -edges of $G \square H$ form a queue in ϕ . Therefore ϕ admits a $(\text{sqn}(G) + \text{qn}(H))$ -queue layout of $G \square H$.

Now we prove the lower bound. By Lemmata 1 and 8(a), $\text{qn}(G \square H) > \eta(G \square H)/2 = (\eta(G) + \eta(H))/2$. The result follows since $\eta(G) \geq \frac{1}{c} \text{sqn}(G)$ and $\eta(H) \geq \frac{1}{c} \text{qn}(H)$. \square

Theorem 2 has the following immediate corollary.

Corollary 1 For all graphs G_1, G_2, \dots, G_d ,

$$\text{qn}(G_1 \square G_2 \square \dots \square G_d) \leq \text{qn}(G_1) + \sum_{i=2}^d \text{sqn}(G_i) .$$

\square

5.1 Grids

A d -dimensional grid is a graph $P_{n_1} \square P_{n_2} \square \dots \square P_{n_d}$, for all $n_i \geq 1$. Heath and Rosenberg [19] determined the queue-number of every 2-dimensional grid.

Lemma 9 ([19]) Every 2-dimensional grid has queue-number one. \square

A generalised d -dimensional grid is a graph $G = P_{n_1}^k \square P_{n_2}^k \square \dots \square P_{n_d}^k$, for all $k \geq 1$ and $n_i \geq k + 1$. Now P_n^k has $kn - k(k + 1)/2$ edges. Thus $\eta(P_n^k) = k - \frac{k(k+1)}{2n}$. By Lemma 8(a),

$$\eta(G) = \sum_{i=1}^d \left(k - \frac{k(k+1)}{2n_i} \right) = dk - \frac{1}{2} k(k+1) \sum_{i=1}^d \frac{1}{n_i} . \quad (4)$$

Lemma 9 generalises as follows.

Theorem 3 For all $d \geq 2$, the queue-number of a d -dimensional grid $G = P_{n_1} \square P_{n_2} \square \cdots \square P_{n_d}$ satisfies:

$$\frac{d}{4} \leq \frac{1}{2} \left(d - \sum_{i=1}^d \frac{1}{n_i} \right) < \text{qn}(G) \leq d - 1 .$$

Proof: The lower bound follows from Lemma 1 and Equation (4) with $k = 1$.

For the upper bound, we have $\text{qn}(P_{n_1} \square P_{n_2}) = 1$ by Lemma 9. Obviously $\text{sqn}(P_{n_i}) = 1$ for all $i \geq 3$. Thus $\text{qn}(G) \leq d - 1$ by Corollary 1.

We now give an alternative proof of the upper bound using a different construction. The graph G can be thought of as having vertex set $\{(x_1, x_2, \dots, x_d) : 1 \leq x_i \leq n_i, 1 \leq i \leq d\}$, where two vertices (x_1, x_2, \dots, x_d) and (y_1, y_2, \dots, y_d) are adjacent if and only if $|x_i - y_i| = 1$ for some i , and $x_j = y_j$ for all $j \neq i$. We say this edge is in the i -th dimension. For all $s \geq 0$, let V_s be the set of vertices

$$V_s = \{(x_1, x_2, \dots, x_d) : \sum_{i=1}^d x_i = s\} .$$

Order the vertices (V_0, V_1, \dots) , where each V_s is ordered lexicographically. If vw is an edge then v and w differ in exactly one coordinate, and $v \in V_s$ and $w \in V_{s+1}$ for some s . Thus if two edges vw and pq are nested then $v, p \in V_s$ and $w, q \in V_{s+1}$ for some s . Let Q_i be the set of edges in the i -th dimension. Consider two edges e and f in Q_i . Say

$$e = (x_1, x_2, \dots, x_d)(x_1, \dots, x_{i-1}, x_i + 1, x_{i+1}, \dots, x_d) ,$$

and

$$f = (y_1, y_2, \dots, y_d)(y_1, \dots, y_{i-1}, y_i + 1, y_{i+1}, \dots, y_d) .$$

Without loss of generality $(x_1, x_2, \dots, x_d) \prec (y_1, y_2, \dots, y_d)$, which implies that

$$(x_1, \dots, x_{i-1}, x_i + j, x_{i+1}, \dots, x_d) \prec (y_1, \dots, y_{i-1}, y_i + j, y_{i+1}, \dots, y_d) .$$

Thus e and f are not nested, and Q_i is a queue. Hence we have a d -queue layout. (At this point we have in fact proved that the lexicographical order admits a d -queue layout.)

We now prove that $Q_{d-1} \cup Q_d$ is a queue, and thus we obtain the claimed $(d-1)$ -queue layout. Suppose two edges $e \in Q_{d-1}$ and $f \in Q_d$ are nested. Say

$$e = (x_1, x_2, \dots, x_d)(x_1, x_2, \dots, x_{d-1} + 1, x_d) ,$$

and

$$f = (y_1, y_2, \dots, y_d)(y_1, y_2, \dots, y_{d-1}, y_d + 1) .$$

Then for some s , both (x_1, x_2, \dots, x_d) and (y_1, y_2, \dots, y_d) are in V_s , and both $(x_1, x_2, \dots, x_{d-1} + 1, x_d)$ and $(y_1, y_2, \dots, y_{d-1}, y_d + 1)$ are in V_{s+1} .

Case 1. $(x_1, x_2, \dots, x_d) \prec (y_1, y_2, \dots, y_d)$: Let j be the first dimension for which $x_j < y_j$. If $j \leq d - 2$ then

$$(x_1, x_2, \dots, x_{d-2}, x_{d-1} + 1, x_d) \prec (y_1, y_2, \dots, y_{d-1}, y_d + 1) ,$$

which implies that e and f are not nested. Observe that $j \neq d$ as (x_1, x_2, \dots, x_d) and (y_1, y_2, \dots, y_d) differ in at least two coordinates, since $\sum_i x_i = \sum_i y_i$. Thus $j = d - 1$. That is,

$$x_{d-1} \leq y_{d-1} - 1 . \quad (5)$$

Since e and f are nested, we have $(y_1, y_2, \dots, y_{d-1}, y_d + 1) \prec (x_1, x_2, \dots, x_{d-2}, x_{d-1} + 1, x_d)$, which implies that $y_{d-1} \leq x_{d-1} + 1$. By Equation (5), $x_{d-1} = y_{d-1} - 1$. Since $x_{d-1} + x_d = y_{d-1} + y_d$, we have $x_d = y_d + 1$, which implies that

$$(y_1, y_2, \dots, y_{d-1}, y_d + 1) = (x_1, x_2, \dots, x_{d-2}, x_{d-1} + 1, x_d) .$$

That is, the right-hand endpoints of e and f are the same vertex. Hence e and f are not nested.

Case 2. $(y_1, y_2, \dots, y_d) \prec (x_1, x_2, \dots, x_d)$: By the same argument employed above, the first coordinate for which (y_1, y_2, \dots, y_d) and (x_1, x_2, \dots, x_d) differ is $d - 1$. That is,

$$y_{d-1} < x_{d-1} . \quad (6)$$

Since e and f are nested, we have $(x_1, x_2, \dots, x_{d-2}, x_{d-1} + 1, x_d) \prec (y_1, y_2, \dots, y_{d-1}, y_d + 1)$. Thus $x_{d-1} + 1 < y_{d-1}$, which contradicts Equation (6). Hence e and f are not nested.

Therefore $Q_1, Q_2, \dots, Q_{d-2}, Q_{d-1} \cup Q_d$ is the desired $(d - 1)$ -queue layout. \square

More generally we have the following.

Theorem 4 *The queue-number of a generalised d -dimensional grid $G = P_{n_1}^k \square P_{n_2}^k \square \dots \square P_{n_d}^k$ (where $n_i \geq k + 1$) satisfies:*

$$\frac{dk}{4} \leq \frac{dk}{2} - \frac{k(k+1)}{4} \sum_{i=1}^d \frac{1}{n_i} < \text{qn}(G) \leq \lceil (d - \frac{1}{2})k \rceil .$$

Proof: By Lemma 6, $\text{qn}(P_n^k) = \lceil \frac{k}{2} \rceil$ and $\text{sqn}(P_n^k) \leq k$. Thus, the upper bound follows from Corollary 1. Thus the lower bound follows from Lemma 1 and Equation (4). \square

By Theorem 4 with $k = n - 1$ we have the following.

Corollary 2 *The queue-number of the d -dimensional Hamming graph $G = K_n \square K_n \square \dots \square K_n$ satisfies:*

$$\frac{d(n-1)}{4} < \text{qn}(G) \leq \lceil (d - \frac{1}{2})(n-1) \rceil .$$

A generalised d -dimensional toroidal grid is a graph $C_{n_1}^k \square C_{n_2}^k \square \dots \square C_{n_d}^k$ for all $k \geq 1$ and $n_i \geq 2k + 1$.

Theorem 5 *The queue-number of a generalised toroidal grid $G = C_{n_1}^k \square C_{n_2}^k \square \dots \square C_{n_d}^k$ (where $n_i \geq 2k + 1$) satisfies:*

$$\frac{kd}{2} < \text{qn}(G) \leq (2d - 1)k .$$

Proof: Since $\eta(G) = kd$, we have that $\text{qn}(G) > \frac{kd}{2}$ by Lemma 1. Thus $\text{qn}(G) \geq \lfloor \frac{d}{2} \rfloor + 1$. By Lemma 7, $\text{qn}(C_{n_1}^k) \leq k$ and $\text{sqn}(C_{n_1}^k) \leq 2k$. By Corollary 1, $\text{qn}(G) \leq 2k(d - 1) + k = (2d - 1)k$ \square

6 Direct and Strong Products

We have the following bounds on the queue-number of direct and strong products.

Theorem 6 For all graphs G and H ,

$$\text{qn}(G \times H) \leq 2 \text{sqn}(G) \cdot \text{qn}(H) .$$

Furthermore, if $\text{sqn}(G) \leq c \cdot \eta(G)$ and $\text{qn}(H) \leq c \cdot \eta(H)$, then

$$\text{qn}(G \times H) > \frac{1}{c^2} \text{sqn}(G) \cdot \text{qn}(H) .$$

Proof: First we prove the upper bound. Let $k := \text{sqn}(G)$, and let $(\sigma, \{Q_1, Q_2, \dots, Q_k\})$ be a strict k -queue layout of G . Let $\ell := \text{qn}(H)$, and let $(\pi, \{P_1, P_2, \dots, P_\ell\})$ be an ℓ -queue layout of H . For $1 \leq i \leq k$ and $1 \leq j \leq \ell$, let

$$E'_{i,j} := \{(v, a)(w, b) \in E(G \times H) : vw \in Q_i, ab \in P_j, \sigma(v) < \sigma(w), \pi(a) < \pi(b)\}$$

$$E''_{i,j} := \{(v, a)(w, b) \in E(G \times H) : vw \in Q_i, ab \in P_j, \sigma(v) < \sigma(w), \pi(b) < \pi(a)\}$$

Then $\{E'_{i,j}, E''_{i,j} : 1 \leq i \leq k, 1 \leq j \leq \ell\}$ is a partition of $E(G \times H)$ into $2k\ell$ sets. Let ϕ be the vertex ordering of $G \times H$ in which $\phi(v, a) < \phi(w, b)$ if and only if $\sigma(v) < \sigma(w)$, or $v = w$ and $\pi(a) < \pi(b)$.

We claim that each set $E'_{i,j}$ and $E''_{i,j}$ is a queue in ϕ .

Suppose that two edges $(v, a)(w, b), (x, c)(y, d) \in E'_{i,j}$ are nested. Without loss of generality, $\phi(v, a) < \phi(x, c) < \phi(y, d) < \phi(w, b)$. If $v \neq x$ and $y \neq w$, then $\sigma(v) < \sigma(x) < \sigma(y) < \sigma(w)$, and the edges $vw, xy \in Q_i$ are nested in σ . If $v \neq x$ and $y = w$, then $\sigma(v) < \sigma(x) < \sigma(y) = \sigma(w)$, and the edges $vw, xy \in Q_i$ overlap in σ . If $v = x$ and $y \neq w$, then $\sigma(v) = \sigma(x) < \sigma(y) < \sigma(w)$, and the edges $vw, xy \in Q_i$ overlap in σ . Each of these outcomes contradict the assumption that Q_i is a strict queue in σ . Otherwise $v = x$ and $y = w$, in which case $\pi(a) < \pi(c) < \pi(d) < \pi(b)$, and ab and cd are nested in π . This contradicts the assumption that P_j is a queue in π . Thus each $E'_{i,j}$ is queue in ϕ . By symmetry, each $E''_{i,j}$ is also a queue in ϕ .

Now we prove the lower bound. Lemmata 1 and 8(b) imply that

$$\text{qn}(G \times H) > \eta(G \times H)/2 = \eta(G) \cdot \eta(H) \geq \frac{1}{c} \text{sqn}(G) \cdot \frac{1}{c} \text{qn}(H) .$$

□

Theorem 7 For all graphs G and H ,

$$\text{qn}(G \boxtimes H) \leq 2 \text{sqn}(G) \cdot \text{qn}(H) + \text{sqn}(G) + \text{qn}(H) .$$

Furthermore, if $\text{sqn}(G) \leq c \cdot \eta(G)$ and $\text{qn}(H) \leq c \cdot \eta(H)$, then

$$\text{qn}(G \boxtimes H) > \frac{1}{c^2} \text{sqn}(G) \cdot \text{qn}(H) + \frac{1}{2c} (\text{sqn}(G) + \text{qn}(H)) .$$

Proof: To prove the upper bound, observe that the vertex ordering ϕ defined in Theorems 2 and 6 is the same. By Theorem 2, ϕ admits a $\text{sqn}(G) + \text{qn}(H)$ -queue layout of $G \square H$. By Theorem 6, ϕ admits a

$2 \text{sqn}(G) \cdot \text{qn}(H)$ -queue layout of $G \times H$. Since $G \boxtimes H = (G \square H) \cup (G \times H)$, ϕ admits the claimed queue layout of $G \boxtimes H$.

For the lower bound, Lemmata 1 and 8(c) imply that

$$\text{qn}(G \boxtimes H) > \frac{1}{2}\eta(G \boxtimes H) = \eta(G) \cdot \eta(H) + \frac{1}{2}(\eta(G) + \eta(H)) \geq \frac{1}{c} \text{sqn}(G) \cdot \frac{1}{c} \text{qn}(H) .$$

□

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