Planar Decompositions and the Crossing Number of Graphs with an Excluded Minor*

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Abstract. Tree decompositions of graphs are of fundamental importance in structural and algorithmic graph theory. Planar decompositions generalise tree decompositions by allowing an arbitrary planar graph to index the decomposition. We prove that every graph that excludes a fixed graph as a minor has a planar decomposition with bounded width and a linear number of bags.

The crossing number of a graph is the minimum number of crossings in a drawing of the graph in the plane. We prove that planar decompositions are intimately related to the crossing number, in the sense that a graph with bounded degree has linear crossing number if and only if it has a planar decomposition with bounded width and linear order. It follows from the above result about planar decompositions that every graph with bounded degree and an excluded minor has linear crossing number.

Analogous results are proved for the convex and rectilinear crossing numbers. In particular, every graph with bounded degree and bounded tree-width has linear convex crossing number, and every $K_{3,3}$-minor-free graph with bounded degree has linear rectilinear crossing number.

1 Introduction

The *crossing number* of a graph G, denoted by $\mathsf{cr}(G)$, is the minimum number of crossings in a drawing[1] of G in the plane; see the survey [16]. Crossing number is an important measure of non-planarity, with applications in discrete and computational geometry, graph visualisation, and VLSI circuit design.

* The full version of this extended abstract is reference [17].

** The research of David Wood is supported by a Marie Curie Fellowship of the European Community under contract 023865, and by the projects MCYT-FEDER BFM2003-00368 and Gen. Cat 2001SGR00224.

[1] A *drawing* of a graph represents each vertex by a distinct point in the plane, and represents each edge by a simple closed curve between its endpoints, such that the only vertices an edge intersects are its own endpoints, and no three edges intersect at a common point (except at a common endpoint). A *crossing* is a point of intersection between two edges (other than a common endpoint). Undefined terminology can be found in the monographs [5, 10].

M. Kaufmann and D. Wagner (Eds.): GD 2006, LNCS 4372, pp. 150–161, 2007.

Upper bounds on the crossing number are the focus of this paper. Obviously $\mathsf{cr}(G) \leq \binom{\|G\|}{2}$ for every graph G, where $|G| := |V(G)|$ and $\|G\| := |E(G)|$. A graph family $\mathcal{F}$ has *linear* crossing number if for some constant c, every graph $G \in \mathcal{F}$ has crossing number $\mathsf{cr}(G) \leq c\,|G|$. For example, Pach and Tóth [11] proved that graphs of bounded genus and bounded degree have linear crossing number. Our main result states that bounded-degree graphs that exclude a fixed graph as a minor have linear crossing number.

Theorem 1. *For every graph H there is a constant $c = c(H)$, such that every H-minor-free graph G has crossing number at most $c\,\Delta(G)^2\,|G|$.*

Theorem 1 implies the above-mentioned result of Pach and Tóth [11], since graphs of bounded genus exclude a fixed graph as a minor (although the dependence on Δ is different in the two proofs; see Section 5). Moreover, there are graphs with a fixed excluded minor and unbounded genus. For other recent work on minors and crossing number see [3, 8].

Note that the assumption of bounded degree in Theorem 1 is unavoidable. For example, the complete bipartite graph $K_{3,n}$ has no K_5-minor, yet has $\Omega(n^2)$ crossing number [12]. Conversely, bounded degree does not by itself guarantee linear crossing number. For example, a random cubic graph on n vertices has $\Omega(n)$ bisection width, which implies that it has $\Omega(n^2)$ crossing number. Also note that $c \leq \frac{20}{3}$ in Theorem 1 with $H = K_5$; see [17]. The proof of Theorem 1 is based on *planar decompositions*, which are introduced in the next section.

2 Graph Decompositions

Let G and D be graphs, such that each vertex of D is a set of vertices of G (called a *bag*). We allow distinct vertices of D to be the same set of vertices in G; that is, $V(D)$ is a multiset. For each vertex v of G, let $D(v)$ be the subgraph of D induced by the bags that contain v. Then D is a *decomposition*[2] of G if:

- $D(v)$ is connected and nonempty for each vertex v of G, and
- $D(v)$ and $D(w)$ touch[3] for each edge vw of G.

Let D be a decomposition of a graph G. The *width* of D is the maximum cardinality of a bag. The *order* of D is the number of bags. D has *linear order* if its order is $\mathcal{O}(|G|)$. If the graph D is a tree, then the decomposition D is a *tree decomposition*. If the graph D is a cycle, then the decomposition D is a *cycle decomposition*. The decomposition D is *planar* if the graph D is planar. The *genus* of the decomposition D is the genus of the graph D.

A decomposition D of a graph G is *strong* if $D(v)$ and $D(w)$ intersect for each edge vw of G. The *tree-width* of G, denoted by $\mathsf{tw}(G)$, is 1 less than the minimum

[2] Decompositions, when D is a tree, were introduced by Robertson and Seymour. Diestel and Kühn [6] first generalised the definition for arbitrary graphs D.

[3] Subgraphs A and B of a graph G *intersect* if $V(A) \cap V(B) \neq \emptyset$, and A and B *touch* if they intersect or $v \in V(A)$ and $w \in V(B)$ for some edge vw of G.

width of a strong tree decomposition of G. For example, a graph has tree-width 1 if and only if it is a forest. Graphs with tree-width 2 (called *series-parallel*) are planar, and are characterised as those graphs with no K_4-minor. Tree-width is particularly important in structural and algorithmic graph theory.

For applications to crossing number, tree decompositions are not powerful enough: even the $n \times n$ planar grid has tree-width n. Lemmas 10 and 11 in Section 4 prove the following theorem, which says that planar decompositions are the right type of decomposition for applications to crossing number.

Theorem 2. *A family of graphs with bounded degree has linear crossing number if and only if every graph in the family has a planar decomposition with bounded width and linear order.*

Every tree T satisfies the Helly property: every collection of pairwise intersecting subtrees of T have a vertex in common. It follows that if a tree T is a strong decomposition of G then every clique of G is contained in some bag of T. Other graphs do not have this property. It will be desirable (for performing k-sums in Section 3) that (non-tree) decompositions have a similar property. We therefore introduce the following definitions.

For $p \geq 0$, a *p-clique* is a clique of cardinality p. A *$(\leq p)$-clique* is a clique of cardinality at most p. For $p \geq 2$, a decomposition D of a graph G is a *p-decomposition* if each $(\leq p)$-clique of G is a subset of some bag of D, or is a subset of the union of two adjacent bags of D. An $\omega(G)$-decomposition of G is called an *ω-decomposition*, where $\omega(G)$ is the maximum cardinality of a clique of G. A p-decomposition D of G is *strong* if each $(\leq p)$-clique of G is a subset of some bag of D. Observe that a (strong) 2-decomposition is the same as a (strong) decomposition, and a (strong) p-decomposition also is a (strong) q-decomposition for all $q \in [2, p]$.

In Section 6, we prove the following theorem, which is one of the main contributions of the paper.

Theorem 3. *For every graph H there is an integer $k = k(H)$, such that every H-minor-free graph G has a planar ω-decomposition of width k and order $|G|$.*

3 Manipulating Decompositions

In this section we describe four tools for manipulating graph decompositions. Our first tool describes the effect of contracting an edge in a decomposition. The elementary proof is in the full paper.

Lemma 4 ([17]). *Suppose that D is a planar (strong) p-decomposition of a graph G with width k. Say XY is an edge of D. Then the decomposition D' obtained by contracting the edge XY into the vertex $X \cup Y$ is a planar (strong) p-decomposition of G with width* $\max\{k, |X \cup Y|\}$. *In particular, if $|X \cup Y| \leq k$ then D' also has width k.*

Lemma 5. *Suppose that a graph G has a (strong) planar p-decomposition D of width k and order at most $c|G|$ for some $c \geq 1$. Then G has a (strong) planar p-decomposition of width $c'k$ and order $|G|$, for some c' depending only on c.*

Proof. Without loss of generality, D is a planar triangulation. By a result of Biedl et al. [1], D has a matching M of at least $\frac{1}{3}|D|$ edges. Applying Lemma 4 to each edge of M, we obtain a (strong) planar p-decomposition of G with width at most $2k$ and order at most $\frac{2}{3}|D|$. By induction, for every integer $i \geq 1$, G has a (strong) planar p-decomposition of width $2^i k$ and order at most $(\frac{2}{3})^i|D|$. With $i := \lceil \log_{3/2} c \rceil$, the assumption that $|D| = c|G|$ implies that G has a (strong) planar p-decomposition of width $2^i k$ and order $|G|$. □

Our second tool describes how two decompositions can be composed.

Lemma 6. *Suppose that D is a (strong) p-decomposition of a graph G with width k, and that J is a decomposition of D with width ℓ. Then G has a (strong) p-decomposition isomorphic to J with width $k\ell$.*

Proof. Let J' be the graph isomorphic to J that is obtained by renaming each bag $Y \in V(J)$ by $Y' := \{v \in V(G) : v \in X \in Y \text{ for some } X \in V(D)\}$. There are at most ℓ vertices $X \in Y$, and at most k vertices $v \in X$. Thus each bag of J' has at most $k\ell$ vertices. First we prove that $J'(v)$ is connected for each vertex v of G. Let A' and B' be two bags of J' that contain v. Let A and B be the corresponding bags in D. Thus $v \in X_1$ and $v \in X_t$ for some bags $X_1, X_t \in V(D)$ such that $X_1 \in A$ and $X_t \in B$ (by the construction of J'). Since $D(v)$ is connected, there is a path $X_1, X_2, \dots, X_t$ in D such that v is in each X_i. In particular, each $X_i X_{i+1}$ is an edge of D. Now $J(X_i)$ and $J(X_{i+1})$ touch in J. Thus there is path in J between any vertex of J that contains X_1 and any vertex of J that contains X_t, such that every bag in the path contains some X_i. In particular, there is a path P in J between A and B such that every bag in P contains some X_i. Let $P' := \{Y' : Y \in P\}$. Then $v \in Y'$ for each bag Y' of P' (by the construction of J'). Thus P' is a connected subgraph of J' that includes A' and B', and v is in every such bag. Therefore $J'(v)$ is connected. In the full paper [17] we prove that for each $(\leq p)$-clique C of G, (a) C is a subset of some bag of J', or (b) C is a subset of the union of two adjacent bags of J'.
Moreover, if D is strong then case (a) always occurs. □

The third tool converts a decomposition into an ω-decomposition with a small increase in the width. A graph G is *d-degenerate* if every subgraph of G has a vertex of degree at most d.

Lemma 7. *Every d-degenerate graph G has a strong ω-decomposition isomorphic to G of width at most $d + 1$.*

Proof. It is well known (and easily proved) that G has an acyclic orientation such that each vertex has indegree at most d. Replace each vertex v by the bag $\{v\} \cup N_G^-(v)$. Every subgraph of G has a *sink*. Thus every clique is a subset of some bag. The set of bags that contain a vertex v are indexed by $\{v\} \cup N_G^+(v)$, which induces a connected subgraph in G. Thus we have a strong ω-decomposition. Each bag has cardinality at most $d + 1$. □

Lemmas 6 and 7 imply:

Lemma 8. *Suppose that D is a decomposition of a d-degenerate graph G of width k. Then G has a strong ω-decomposition isomorphic to D of width $k(d+1)$.*

Our fourth tool describes how to determine a planar decomposition of a clique-sum of two graphs, given planar decompositions of the summands[4]. Let G_1 and G_2 be disjoint graphs. Suppose that C_1 and C_2 are k-cliques of G_1 and G_2 respectively, for some integer $k \geq 0$. Let $C_1 = \{v_1, v_2, \dots, v_k\}$ and $C_2 = \{w_1, w_2, \dots, w_k\}$. Let G be a graph obtained from $G_1 \cup G_2$ by identifying v_i and w_i for each $i \in [1, k]$, and deleting an arbitrary (possibly empty) subset of the edges between vertices in C_1 $(= C_2)$. Then G is a *k-sum* of G_1 and G_2. An ℓ-sum for some $\ell \leq k$ is called a *$(\leq k)$-sum*. For example, if G_1 and G_2 are planar then it is easily seen that every (≤ 2)-sum of G_1 and G_2 is also planar.

Lemma 9. *Suppose that for integers $p \leq q$, a graph G is a $(\leq p)$-sum of graphs G_1 and G_2, and each G_i has a (strong) planar q-decomposition D_i of width k_i. Then G has a (strong) planar q-decomposition of width $\max\{k_1, k_2\}$ and order $|D_1| + |D_2|$.*

Proof. Let $C := V(G_1) \cap V(G_2)$. Then C is a $(\leq p)$-clique, and thus a $(\leq q)$-clique, of both G_1 and G_2. Thus for each i, (1) $C \subseteq X_i$ for some bag X_i of D_i, or (2) $C \subseteq X_i \cup Y_i$ for some edge X_iY_i of D_i. If (1) is applicable, which is the case if D_i is strong, then consider $Y_i := X_i$ in what follows.

Let D be the graph obtained from the disjoint union of D_1 and D_2 by adding edges X_1X_2, X_1Y_2, Y_1X_2, and Y_1Y_2. By considering X_1Y_1 to be on the outerface of G_1 and X_2Y_2 to be on the outerface of G_2, observe that D is planar, as illustrated in Figure 1.

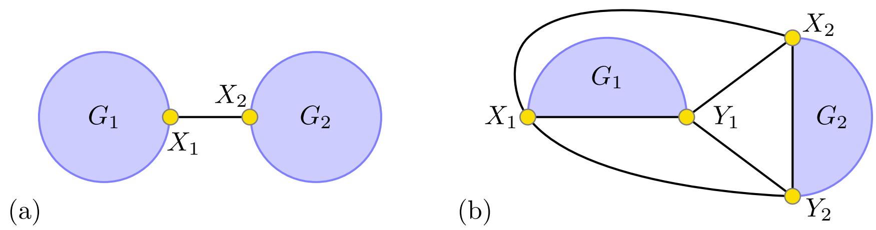

Fig. 1. Sum of (a) strong planar decompositions, (b) planar decompositions

We now prove that $D(v)$ is connected for each vertex v of G. If $v \notin V(G_1)$ then $D(v) = D_2(v)$, which is connected. If $v \notin V(G_2)$ then $D(v) = D_1(v)$, which

[4] Leaños and Salazar [9] recently proved related results on the additivity of crossing numbers.

is connected. Otherwise, $v \in C$. Thus $D(v) = D_1(v) \cup D_2(v)$. Since $v \in X_1 \cup Y_1$ and $v \in X_2 \cup Y_2$, and X_1, Y_1, X_2, Y_2 induce a connected subgraph ($\subseteq K_4$) in D, we have that $D(v)$ is connected.

Each ($\leq q$)-clique B of G is a ($\leq q$)-clique of G_1 or G_2. Thus B is a subset of some bag of D, or B is a subset of the union of two adjacent bags of D. Moreover, if D_1 and D_2 are both strong, then B is a subset of some bag of D. Therefore D is a q-decomposition of G, and if D_1 and D_2 are both strong then D is also strong. The width and order of D are obviously as claimed. □

4 Planar Decompositions and the Crossing Number

The following lemma is the key link between planar decompositions and the crossing number of a graph.

Lemma 10. *Suppose that D is a planar decomposition of a graph G of width k. Then the crossing number of G satisfies*

$$\mathsf{cr}(G) \leq 2\,\Delta(G)^2 \sum_{X \in V(D)} \binom{|X|+1}{2} \leq k(k+1)\,\Delta(G)^2\,|D| \ .$$

Proof. Fix a straight-line drawing of D with no crossings. Let $\epsilon > 0$. Let $R_\epsilon(X)$ be the open disc of radius ϵ centred at each vertex X in the drawing of D. For each edge XY of D, let $R_\epsilon(XY)$ be the union of all segments with one endpoint in $R_\epsilon(X)$ and one endpoint in $R_\epsilon(Y)$. For some $\epsilon > 0$, $R_\epsilon(X) \cap R_\epsilon(Y) = \emptyset$ for all distinct bags X and Y of D, and $R_\epsilon(XY) \cap R_\epsilon(AB) = \emptyset$ for all edges XY and AB of D that have no endpoint in common.

For each vertex v of G, choose a bag S_v of D that contains v. For each vertex v of G, choose a point $p(v) \in R_\epsilon(S_v)$, and for each bag X of D, choose a set $P(X)$ of $\sum_{v \in X} \deg_G(v)$ points in $R_\epsilon(X)$, so that no two points coincide, no three points are collinear, and no three segments, each connecting two points, cross at a common point. These points can be chosen iteratively since each disc $R_\epsilon(X)$ is 2-dimensional[5], but the set of excluded points is 1-dimensional.

Draw each vertex v at $p(v)$. For each edge vw of G, a simple polyline $L(vw) = (p(v), x_1, x_2, \dots, x_a, y_1, y_2, \dots, y_b, p(w))$, defined by its endpoints and bends, is a *feasible* representation of vw if:

(1) each bend x_i is in $P(X_i)$ for some bag X_i containing v,
(2) each bend y_i is in $P(Y_i)$ for some bag Y_i containing w,
(3) the bags $S_v, X_1, X_2, \dots, X_a, Y_1, Y_2, \dots, Y_b, S_w$ are distinct (unless $S_v = S_w$ in which case $a = b = 0$), and
(4) consecutive bends in $L(vw)$ occur in adjacent bags of D.

Since $D(v)$ and $D(w)$ touch, there is a feasible polyline that represents vw.

[5] Let Q be a nonempty set of points in the plane. Then Q is 2-*dimensional* if it contains a disk of positive radius; Q is 1-*dimensional* if it is not 2-dimensional but contains a finite curve; otherwise Q is 0-*dimensional*.

A drawing of G is *feasible* if every edge of G is represented by a feasible polyline, and no two bends coincide. Since each $|P(X)| = \sum_{v \in X} \deg(v)$, there is a feasible drawing. In particular, no edge passes through a vertex and no three edges have a common crossing point.

By properties (1)–(4), each segment in a feasible drawing is contained within $R_\epsilon(X)$ for some bag X of D, or within $R_\epsilon(XY)$ for some edge XY of D. Consider a crossing in G between edges vw and xy. Since D is drawn without crossings, the crossing point is contained within $R_\epsilon(X)$ for some bag X of D, or within $R_\epsilon(XY)$ for some edge XY of D. Thus some endpoint of vw, say v, and some endpoint of xy, say x, are in a common bag X. In this case, charge the crossing to the 5-tuple (vw, v, xy, x, X).

At most four crossings are charged to each 5-tuple (vw, v, xy, x, X), since by property (4), each of vw and xy have at most two segments that intersect $R_\epsilon(X)$ (which might pairwise cross). We prove in [17] that in a feasible drawing that minimises the total (Euclidean) length of the edges (with $\{p(v) : v \in V(G)\}$ and $\{P(X) : X \in V(D)\}$ fixed), at most two crossings are charged to each such 5-tuple. Thus the number of crossings is at most twice the number of 5-tuples. Therefore the number of crossings is at most

$$2 \sum_{X \in V(D)} \sum_{v,x \in X} \deg_G(v) \cdot \deg_G(x) \ \leq \ 2\,\Delta(G)^2 \sum_{X \in V(D)} \binom{|X|+1}{2} \ . \qquad \square$$

Note that the bound on the crossing number in Lemma 10 is within a constant factor of optimal for the complete graph [17]. The following converse result to Lemma 10 is proved by replacing each crossing by a bag.

Lemma 11 ([17]). *Every graph G has a planar decomposition of width* 2 *and order* $|G| + \mathsf{cr}(G)$.

5 Graphs Embedded in a Surface

Let $\mathbb{S}_\gamma$ be the orientable surface with $\gamma \geq 0$ handles. A *cycle* in $\mathbb{S}_\gamma$ is a closed curve in the surface. A cycle is *contractible* if it is contractible to a point in the surface. A noncontractible cycle is *separating* if it separates $\mathbb{S}_\gamma$ into two connected components.

The (*orientable*) *genus* of a graph G is the minimum γ such that G has a 2-cell embedding in $\mathbb{S}_\gamma$. Let G be a graph embedded in $\mathbb{S}_\gamma$. In what follows, by a *face* we mean the set of vertices on the boundary of the face. Let $F(G)$ be the set of faces in G. A *noose* of G is a cycle C in $\mathbb{S}_\gamma$ that does not intersect the interior of an edge of G. Let $V(C)$ be the set of vertices of G intersected by C. The *length* of C is $|V(C)|$.

Pach and Tóth [11] proved that, for some constant c_γ, the crossing number of every graph G of genus γ satisfies

$$\mathsf{cr}(G) \ \leq \ c_\gamma \sum_{v \in V(G)} \deg(v)^2 \ \leq \ 2c_\gamma\,\Delta(G)\,\|G\| \ . \qquad (1)$$

The constant c_γ was subsequently improved by Djidjev and Vrťo [7]. It is well known [17] that $\|G\| \leq (\sqrt{3\gamma}+3)|G|-6$. Thus

$$\mathsf{cr}(G) \ \leq \ c_\gamma \, \Delta(G) \, |G| \ . \tag{2}$$

By Lemma 11, G has a planar decomposition of width 2 and order $c_\gamma \, \Delta(G) \, |G|$. We now provide an analogous result without the dependence on $\Delta(G)$, but at the expense of an increased bound on the width.

Theorem 12. *Every graph G with genus γ has a planar decomposition of width 2^γ and order $3^\gamma|G|$.*

The key to the proof of Theorem 12 is the following lemma, whose proof is inspired by similar ideas of Pach and Tóth [11].

Lemma 13. *Let G be a graph with a 2-cell embedding in $\mathbb{S}_\gamma$ for some $\gamma \geq 1$. Then G has a decomposition of width 2, genus at most $\gamma-1$, and order $3|G|$.*

Proof. Since $\gamma \geq 1$, G has a noncontractible nonseparating noose. Let C be a noncontractible nonseparating noose of minimum length $k := |V(C)|$. Orient C and let $V(C) := (v_1, v_2, \dots, v_k)$ in the order around C. For each vertex $v_i \in V(C)$, let $E^\ell(v_i)$ and $E^r(v_i)$ respectively be the set of edges incident to v_i that are on the left-hand side and right-hand side of C (with respect to the orientation). Cut the surface along C, and attach a disk to each side of the cut. Replace each vertex $v_i \in V(C)$ by two vertices v_i^ℓ and v_i^r respectively incident to the edges in $E^\ell(v_i)$ and $E^r(v_i)$. Embed v_i^ℓ on the left-hand side of the cut, and embed v^r on the right-hand side of the cut. We obtain a graph G' embedded in a surface of genus at most $\gamma-1$ (since C is nonseparating).

Let $L := \{v_i^\ell : v \in V(C)\}$ and $R := \{v_i^r : v \in V(C)\}$. By Menger's Theorem, the maximum number of disjoint paths between L and R in G' equals the minimum number of vertices that separate L from R in G'. Let Q be a minimum set of vertices that separate L from R in G'. Then there is a noncontractible nonseparating noose in G that only intersects vertices in Q. (It is nonseparating in G since L and R are identified in G.) Thus $|Q| \geq k$ by the minimality of $|V(C)|$. Hence there exist k disjoint paths $P_1, P_2, \dots, P_k$ between L and R in G', where the endpoints of P_i are v_i^ℓ and $v_{\sigma(i)}^r$, for some permutation σ of $[1, k]$. In the disc with R on its boundary, draw an edge from each vertex $v_{\sigma(i)}^r$ to v_i^r such that no three edges cross at a single point and every pair of edge cross at most once. Add a new vertex $x_{i,j}$ on each crossing point between edges $v_{\sigma(i)}^r v_i^r$ and $v_{\sigma(j)}^r v_j^r$. Let G'' be the graph obtained. Then G'' is embedded in $S_{\gamma-1}$.

We now make G'' a decomposition of G. Replace v_i^ℓ by $\{v_i\}$ and replace v_i^r by $\{v_i\}$. Replace every other vertex v of G by $\{v\}$. Replace each 'crossing' vertex $x_{i,j}$ by $\{v_i, v_j\}$. Now for each vertex $v_i \in V(C)$, add v_i to each bag on the path P_i from v_i^ℓ to $v_{\sigma(i)}^r$. Thus $G''(v_i)$ is a (connected) path. Clearly $G''(v)$ and $G''(w)$ touch for each edge vw of G. Hence G'' is a decomposition of G with genus at most $\gamma-1$. Since the paths $P_1, P_2, \dots, P_k$ are pairwise disjoint, the width of the decomposition is 2.

It remains to bound the order of G''. Let $n := |G|$. Observe that G'' has at most $n+k+\binom{k}{2}$ vertices. One of the paths P_i has at most $\frac{n+k}{k}$ vertices. For ease of counting, add a cycle to G' around R. Consider the path in G' that starts at v_i^ℓ, passes through each vertex in P_i, and then takes the shortest route from $v^r_{\sigma(i)}$ around R back to v_i^r. The distance between $v^r_{\sigma(i)}$ and v_i^r around R is at most $\frac{k}{2}$. This path in G' forms a noncontractible nonseparating noose in G (since if two cycles in a surface cross in exactly one point, then both are noncontractible).

The length of this noose in G is at most $\frac{n+k}{k} - 1 + \frac{k}{2}$ (since v_i^ℓ and v_i^r both appeared in the path). Hence $\frac{n+k}{k} - 1 + \frac{k}{2} \geq k$ by the minimality of $|V(C)|$. Thus $k \leq \sqrt{2n}$. Therefore G'' has at most $n + \sqrt{2n} + \binom{\sqrt{2n}}{2} \leq 3n$ vertices. □

Proof of Theorem 12. We proceed by induction on γ. If $\gamma = 0$ then G is planar, and G itself is a planar decomposition of width $1 = 2^0$ and order $n = 3^0 n$. Otherwise, by Lemma 13, G has a decomposition D of width 2, genus $\gamma - 1$, and order $3n$. By induction, D has a planar decomposition of width $2^{\gamma-1}$ and order $3^{\gamma-1}(3n) = 3^\gamma n$. By Lemma 6 with $p = k = 2$, and $\ell = 2^{\gamma-1}$, G has a planar decomposition of width $2 \cdot 2^{\gamma-1} = 2^\gamma$ and order $3^\gamma n$. □

Theorem 12 and Lemma 10 imply that every graph G with genus γ has crossing number $\mathsf{cr}(G) \leq 12^\gamma \,\Delta(G)^2\, |G|$, which for fixed γ, is weaker than the bound of Pach and Tóth [11] in (2). The advantage of our approach is that it generalises for graphs with an arbitrary excluded minor.

6 *H*-Minor-Free Graphs

For integers $h \geq 1$ and $\gamma \geq 0$, Robertson and Seymour [13] defined a graph G to be *h-almost embeddable* in $\mathbb{S}_\gamma$ if G has a set X of at most h vertices such that $G \setminus X$ can be written as $G_0 \cup G_1 \cup \cdots \cup G_h$ such that:

- G_0 has an embedding in $\mathbb{S}_\gamma$,
- the graphs $G_1, G_2, \ldots, G_h$ (called *vortices*) are pairwise disjoint,
- there are faces $F_1, F_2, \ldots, F_h$ of the embedding of G_0 in $\mathbb{S}_\gamma$, such that each $F_i = V(G_0) \cap V(G_i)$,
- if $F_i = (u_{i,1}, u_{i,2}, \ldots, u_{i,|F_i|})$ in clockwise order about the face, then G_i has a strong $|F_i|$-cycle decomposition Q_i of width h, such that each vertex $u_{i,j}$ is in the j-th bag of Q_i.

The following 'characterisation' of H-minor-free graphs is a deep theorem by Robertson and Seymour [13].

Theorem 14 ([13]). *For every graph H there is a positive integer $h = h(H)$, such that every H-minor-free graph G can be obtained by $(\leq h)$-sums of graphs that are h-almost embeddable in some surface in which H cannot be embedded.*

Lemma 15. *Every graph G that is h-almost embeddable in $\mathbb{S}_\gamma$ has a planar decomposition of width $h(2^\gamma + 1)$ and order $3^\gamma\,|G|$.*

Proof. By Theorem 12, G_0 has a planar decomposition D of width at most 2^γ and order $3^\gamma |G_0| \leq 3^\gamma |G|$. We can assume that D is connected. For each vortex G_i, add each vertex in the j-th bag of Q_i to each bag of D that contains $u_{i,j}$. The bags of D now contain at most $2^\gamma h$ vertices. Now add X to every bag. The bags of D now contain at most $(2^\gamma + 1)h$ vertices. For each vertex v that is not in a vortex, $D(v)$ is unchanged by the addition of the vortices, and is thus connected. For each vertex v in a vortex G_i, $D(v)$ is the subgraph of D induced by the bags (in the decomposition of G_0) that contain $u_{i,j}$, where v is in the j-th bag of Q_i. Now $Q_i(v)$ is a connected subgraph of the cycle Q_i, and for each vertex $u_{i,j}$, the subgraphs $G_0(u_{i,j})$ and $G_0(u_{i,j+1})$ touch. Thus $D(v)$ is connected. (This argument is similar to that used in Lemma 6.) $D(v)$ is connected for each vertex $v \in X$ since D itself is connected. □

Lemma 16. *For all integers $h \geq 1$ and $\gamma \geq 0$ there is a constant $d = d(h, \gamma)$, such that every graph G that is h-almost embeddable in $\mathbb{S}_\gamma$ is d-degenerate.*

Proof. If G is h-almost embeddable in $\mathbb{S}_\gamma$ then every subgraph of G is h-almost embeddable in $\mathbb{S}_\gamma$. Thus it suffices to prove that if G has n vertices and m edges, then its average degree $\frac{2m}{n} \leq d$. Say each G_i has m_i edges. G has at most hn edges incident to X. Thus $m \leq hn + \sum_{i=0}^{h} m_i$. Now $m_0 < (\sqrt{3\gamma} + 3)n$. Both endpoints of an edge of a vortex G_i is in some bag of Q_i. Thus $m_i \leq \binom{h}{2}|F_i|$. Since $G_1, G_2, \dots, G_h$ are pairwise disjoint, $\sum_{i=1}^{h} m_i \leq \binom{h}{2}n$. Thus $m < (h + \sqrt{3\gamma} + 3 + \binom{h}{2})n$. Taking $d = h(h+1) + 2\sqrt{3\gamma} + 6$ we are done. □

Lemmas 8, 15 and 16 imply:

Corollary 17. *For all integers $h \geq 1$ and $\gamma \geq 0$ there is a constant $k = k(h, \gamma) \geq \gamma$, such that every graph G that is h-almost embeddable in $\mathbb{S}_\gamma$ has a planar ω-decomposition of width k and order $3^\gamma |G|$.* □

Now we bring in $(\leq h)$-sums.

Lemma 18. *For all integers $h \geq 1$ and $\gamma \geq 0$, every graph G that can be obtained by $(\leq h)$-sums of graphs that are h-almost embeddable in $\mathbb{S}_\gamma$ has a planar ω-decomposition of width k and order* $\max\{1, 3^\gamma(h+1)(|G| - h)\}$*, where $k = k(h, \gamma)$ from Corollary 17.*

Proof. If $|G| \leq h$ then the decomposition of G with all its vertices in a single bag satisfies the claim (since $k \geq h$). Now assume that $|G| \geq h + 1$. If G is h-almost embeddable in $\mathbb{S}_\gamma$, then by Corollary 17, G has a planar ω-decomposition of width k and order $3^\gamma|G| \leq 3^\gamma(h+1)(|G| - h)$. Otherwise, G is a $(\leq h)$-sum of graphs G_1 and G_2, each of which, by induction, has a planar ω-decomposition of width k and order $\max\{1, 3^\gamma(h+1)(|G_i| - h)\}$. By Lemma 9, G has a planar ω-decomposition D of width k and order $|D| = \max\{1, 3^\gamma(h+1)(|G_1| - h)\} + \max\{1, 3^\gamma(h+1)(|G_2| - h)\}$. Without loss of generality, $|G_1| \leq |G_2|$. If $|G_2| \leq h$ then $|D| = 2 \leq 3^\gamma(h+1)(|G| - h)$. If $|G_1| \leq h$ and $|G_2| \geq h + 1$, then $|D| = 1 + 3^\gamma(h+1)(|G_2| - h) \leq 3^\gamma(h+1)(|G| - h)$. Otherwise, both $|G_1| \geq h + 1$ and $|G_2| \geq h + 1$. Thus $|D| \leq 3^\gamma(h+1)(|G_1| + |G_2| - 2h) \leq 3^\gamma(h+1)(|G| - h)$. □

Proof of Theorem 3. Let $h = h(H)$ from Theorem 14. Let $\mathbb{S}_\gamma$ be the surface in Theorem 14 in which H cannot be embedded. By Theorem 14, G can be obtained by $(\leq h)$-sums of graphs that are h-almost embeddable in $\mathbb{S}_\gamma$. By Lemma 18, G has a planar ω-decomposition of width k and order $3^\gamma(h+1)|G|$, where $k = k(h,\gamma)$ from Corollary 17. By Lemma 5, G has a planar ω-decomposition of width k' and order $|G|$, for some k' only depending on k, γ and h. □

Observe that Lemma 10 and Theorem 3 prove Theorem 1.

7 Complementary Results

A graph drawing is *rectilinear* (or *geometric*) if each edge is represented by a straight line-segment. The *rectilinear crossing number* of a graph G, denoted by $\overline{\mathsf{cr}}(G)$, is the minimum number of crossings in a rectilinear drawing of G; see [2, 14]. A rectilinear drawing is *convex* if the vertices are positioned on a circle. The *convex* (or *outerplanar*) *crossing number* of a graph G, denoted by $\mathsf{cr}^\star(G)$, is the minimum number of crossings in a convex drawing of G; see [4, 15]. Obviously $\mathsf{cr}(G) \leq \overline{\mathsf{cr}}(G) \leq \mathsf{cr}^\star(G)$ for every graph G. *Linear* rectilinear and *linear* convex crossing numbers are defined in an analogous way to linear crossing number.

It is unknown whether an analogue of Theorem 1 holds for rectilinear crossing number[6]. On the other hand, we prove such a result for $K_{3,3}$-minor-free graphs.

Theorem 19 ([17]). *Every $K_{3,3}$-minor-free graph G has a rectilinear drawing in which each edge crosses at most $2\,\Delta(G)$ other edges. Hence $\overline{\mathsf{cr}}(G) \leq \Delta(G)\,\|G\| \leq \Delta(G)\,(3\,|G| - 5)$.*

An analogue of Theorem 1 for convex crossing number does not hold, even for planar graphs, since Shahrokhi et al. [15] proved that the $n \times n$ planar grid G_n (which has maximum degree 4) has convex crossing number $\Omega(|G_n| \log |G_n|)$. Now, G_n has tree-width n. In the following sense, we prove that large tree-width necessarily forces up the convex crossing number.

Theorem 20 ([17]). *Every graph G with degree at most Δ and tree-width at most k has a convex drawing in which each edge crosses $\mathcal{O}(k\Delta^2)$ other edges. Hence $\mathsf{cr}^\star(G) \leq \mathcal{O}(k\Delta^2\|G\|) \leq \mathcal{O}(k^2\Delta^2|G|)$. Conversely, suppose that a graph G has a convex drawing such that whenever two edges e and f cross, e or f crosses at most ℓ edges. Then G has tree-width at most $3\ell + 11$.*

In particular, graphs of bounded degree and bounded tree-width have linear convex crossing number. Again, the assumption of bounded degree is necessary since $K_{3,n}$ has tree-width 3 and crossing number $\Omega(n^2)$.

[6] The crossing number and rectilinear crossing number are not related in general. In particular, for every integer $k \geq 4$, Bienstock and Dean [2] constructed a graph G_k with crossing number 4 and rectilinear crossing number k. It is easily seen that G_k has no K_{14}-minor. However, the maximum degree of G_k increases with k. Thus G_k is not a counterexample to an analogue of Theorem 1 for rectilinear crossing number.

Acknowledgements. Thanks to János Pach and Géza Tóth for explaining their proofs in reference [11].

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