1. Let $M$ denote the symmetric $4 \times 4$ matrix given by $M = \begin{bmatrix} P & O \\ O & Q \end{bmatrix}$, where $P = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$, $Q = \begin{bmatrix} d & f \\ f & e \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(a) Show that $M^{-1} = \begin{bmatrix} P^{-1} & O \\ O & Q^{-1} \end{bmatrix}$, if the inverse exists.

(b) Find the inverse of each of the following matrices:

i. \begin{bmatrix} 7 & 2 \\ 2 & 2 \end{bmatrix};

ii. \begin{bmatrix} 10 & 0 & 0 \\ 0 & 7 & 2 \\ 0 & 2 & 2 \end{bmatrix};

iii. \begin{bmatrix} 7 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 6 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix};

iv. \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 7 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}.

2. $Y_1, Y_2, \ldots, Y_8$ are independent normal random variables with constant variance, $\sigma^2$, and expectations given by:

$E(Y_1) = \alpha + \beta, \ E(Y_2) = \beta + \gamma, \ E(Y_3) = \alpha + \gamma, \ E(Y_4) = \alpha - \beta + \gamma,$

$E(Y_5) = \alpha - \beta, \ E(Y_6) = \beta - \gamma, \ E(Y_7) = \alpha - \gamma, \ E(Y_8) = \alpha + \beta - \gamma.$

Observations give:

$y_1 = 19.2, y_2 = 9.6, y_3 = 15.5, y_4 = 9.4, y_5 = 6.5, y_6 = 2.9, y_7 = 7.0, y_8 = 15.1.$

(a) Write down the observational equations.

(b) Derive the normal equations.

(c) Obtain the least squares estimates of $\alpha$, $\beta$ and $\gamma$. Specify their standard errors, and give 95% confidence intervals for $\alpha$, $\beta$ and $\gamma$.

(d) Obtain an estimate of $\sigma^2$ and give a 95% confidence interval for $\sigma^2$.

(e) Carry out a test of $H_0: \alpha = \beta + \gamma$.

(f) Find a 95% confidence interval for $\alpha + \beta + \gamma$.

(g) Find a 95% prediction interval for an observation on $Y_9$, which is such that $E(Y_9) = \alpha + \beta + \gamma$ and $\text{var}(Y_9) = \sigma^2$.

(h) Use MINITAB command regr to check your answers.

---

**Addition problems 10**

MINITAB is not particularly clever with matrices, but it can be used for basic matrix calculations:

MTB > READ 3 3 M23
DATA> 1 2 3
DATA> 4 5 6
DATA> 7 8 9

puts the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ into M23. (Matrices are stored in M1, M2, \ldots). To see it, PRINT M23.

The command MULTIPLY M1 M2 M3 calculates the product M1*M2 (if the matrices are compatible for multiplication) and puts the result into M3. The command ADD M1 M2 M3 operates similarly for addition. The command TRANSPOSE M1 M2 puts the transpose of M1 into M2. The command INVERT M1 M2 puts the inverse of M1 (if it exists) into M2. And to see the inverse, PRINT M2.
4. Use the matrix commands to find \((A'A)^{-1}\): (a) if \(A = \begin{bmatrix} 1 & 2 \\ 1 & 6 \\ 1 & 3 \end{bmatrix}\); (b) if \(A = \begin{bmatrix} 1 & 3 \\ 1 & 5 \\ 1 & 3 \end{bmatrix}\).

5. Fit the model \(E(Y \mid x) = \alpha + \beta x\), \(\text{var}(Y \mid x) = \sigma^2\) to the following data:

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>13</td>
</tr>
<tr>
<td>6</td>
<td>23</td>
</tr>
<tr>
<td>3</td>
<td>19</td>
</tr>
<tr>
<td>5</td>
<td>22</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
</tr>
</tbody>
</table>

i.e. find estimates of \(\alpha\), \(\beta\) and \(\sigma^2\).

(a) using the matrix commands;
(b) using the \texttt{REGR} command.

6. Fit the model \(E(Y \mid x_1, x_2) = \alpha + \beta_1 x_1 + \beta_2 x_2\), \(\text{var}(Y \mid x_1, x_2) = \sigma^2\) to the following data:

<table>
<thead>
<tr>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>13</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>19</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>22</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
<td>15</td>
</tr>
</tbody>
</table>

i.e. find estimates of \(\alpha\), \(\beta_1\), \(\beta_2\) and \(\sigma^2\).

(a) using the matrix commands;
(b) using the \texttt{REGR} command.

Clearly the matrix commands are inefficient for standard problems — but they do indicate what the built-in routines are actually doing.

Even non-standard linear model problems can be effectively handled on \textsc{minitab} by entering the columns of \(A\) into \texttt{c11, c12, \ldots, c1p}; and then, with \(y\) in \texttt{c10}, the command \texttt{regr c10 p c11 - c1p; noconstant}. (with an appropriate numerical value for \(p\) of course), will fit the model \(y = A\theta + \varepsilon\), including the anova in the form \(y' y = \theta' A' y + d'd\). Confidence intervals and prediction intervals can also be obtained using the subcommand \texttt{predict c1 \ldots c1p;} this gives a confidence interval for \(\eta = c' \theta\) and a prediction interval for an observation with mean \(\eta\) under the linear model assumptions.

7. Use \textsc{minitab} to check your answers for problems 8.4, 8.5, 8.6 and 8.7.

If the \texttt{noconstant} subcommand is omitted, then \textsc{minitab} assumes the first column of \(A\) is a column of 1s, corresponding to an overall mean; and the analysis is done in terms of deviations from the overall mean. In particular the total sum of squares is corrected for the mean: it is then the more commonly used \(\sum(y - \bar{y})^2 = \sum y^2 - n \bar{y}^2\) with \(df = n - 1\). The term \(n \bar{y}^2\) is the sum of squares due to the mean, and subtracting it is referred to as correcting for the mean.

8. \(Y_1, Y_2, \ldots, Y_6\) are independent normal random variables with constant variance, \(\sigma^2\), and expectations given by:

- \(E(Y_1) = \alpha\), \(E(Y_2) = \beta\), \(E(Y_3) = \alpha + \beta\),
- \(E(Y_4) = \alpha - \beta\), \(E(Y_5) = 2\alpha + \beta\), \(E(Y_6) = \alpha + 2\beta\)

Observations give:

\[ y_1 = 7, \ y_2 = 1, \ y_3 = 2, \ y_4 = 9, \ y_5 = 10, \ y_6 = 5 \]

(a) Write down the observational equations.
(b) Write down the normal equations.
(c) Obtain the least squares estimates of \(\alpha\) and \(\beta\), specify their standard errors, and give 95% confidence intervals for \(\alpha\) and \(\beta\).
(d) Obtain an estimate of \(\sigma^2\) and give a 95% confidence interval for \(\sigma^2\).
(e) Carry out a test of \(H_0: \alpha + 3\beta = 0\).
(f) Find a 95% confidence interval for \(\alpha - 2\beta\).
(g) Find a 95% prediction interval for an observation on \(Y_7\), which is such that \(E(Y_7) = \alpha - 2\beta\) and \(\text{var}(Y_7) = \sigma^2\).