Tutorial problems for next week’s tutorial: 4.3, 4.6, 4.8, 4.9, 4.11.

Solutions to problems 1–4 are to handed in for assessment.

1. A random sample of \( n = 33 \) on \( X \overset{d}{=} \text{N}(\mu, \sigma^2) \) yields the following results:

\[
\begin{array}{cccccccccccccc}
45.9 & 55.2 & 42.3 & 30.1 & 52.7 & 49.1 & 70.6 & 47.3 & 49.8 & 65.5 \\
56.5 & 50.2 & 57.1 & 45.0 & 57.0 & 58.6 & 59.3 & 50.5 & 52.2 & 53.3 \\
43.1 & 44.5 & 67.6 & 63.4 & 71.4 & 42.8 & 31.5 & 76.3 & 67.2 & 56.9 \\
\end{array}
\]

(a) Test \( \mu = 50 \) against a two-sided alternative, specifying the \( P \)-value.

(b) Test \( \sigma = 10 \) against a two-sided alternative, specifying the \( P \)-value.

2. A random sample of \( n \) observations is obtained from a population distribution that depends on a parameter \( \theta \). It is known that the statistic \( U_n \) is such that

\[
U_n \overset{d}{=} \text{N}(\theta, \theta(1 + \theta) n) \quad \text{as} \quad N \to \infty.
\]

We have a sample of \( n = 100 \) for which \( u_{\text{obs}} = 5.94 \). Use the above asymptotic result to test \( H_0: \theta = 5 \) against \( H_1: \theta > 5 \).

A second statistic \( V_n \) is such that

\[
V_n \overset{d}{=} \text{N}(\theta(1 + \theta), \theta n) \quad \text{as} \quad N \to \infty.
\]

For a sample of \( n = 100 \), \( v_{\text{obs}} = 30.94 \). Use this observation to test \( H_0: \theta = 5 \) against \( H_1: \theta > 5 \).

3. It is required to test \( H_0: \mu = 50 \) based on a sample of 10 observations from \( X \overset{d}{=} \text{N}(\mu, 10^2) \): find \( c \) so that the decision rule “reject \( H_0 \) if \( \bar{X} > c \)” has size 0.05. Sketch a graph of the power of this test for \( 50 \leq \mu \leq 60 \).

Repeat this procedure for a sample of 100 observations. Sketch the power curves (for \( n = 10 \) and \( n = 100 \)) on the same graph.

4. It is required to test \( H_0: p = 0.4 \) vs \( H_1: p < 0.4 \) for a sequence of Bernoulli trials. If the test is to have size at most 0.05 and power at least 0.9 when \( p = 0.3 \), how big a sample is required?

5. To test \( H_0: X \overset{d}{=} \text{N}(10, 5^2) \), against \( H_1: X \overset{d}{=} \text{N}(15, 5^2) \), a random sample of \( n = 16 \) observations is obtained on \( X \); and the decision rule is to reject \( H_0 \) if \( \bar{X} > 12 \). The observation \( \bar{x} = 12.73 \) is obtained. Evaluate

(i) the size; (ii) the power; (iii) the \( P \)-value.

6. If 120 trials yield 46 successes, does this represent significant evidence that the probability of success is less than 0.5?

State this as a hypothesis testing problem, specifying precisely the null and alternative hypotheses, the test statistic and its distribution.

7. A random sample of \( n = 40 \) on \( X \overset{d}{=} \text{N}(\mu, \sigma^2) \) yields \( \bar{x} = 47.69 \) and \( s = 7.43 \).

(a) Test \( \mu = 50 \) against a two-sided alternative, specifying the \( P \)-value.

(b) Test \( \sigma = 5 \) against a two-sided alternative, specifying the \( P \)-value.

8. It is required to test \( H_0: \lambda = 10 \) vs \( H_1: \lambda > 10 \) for a Poisson distributed population. If the test is to have size at most 0.05 and power at least 0.9 when \( \lambda = 11 \), how big a sample is required?
9. [cf. HW=3.3] The following is a random sample on $Y \sim N(\mu, \sigma^2)$:

49.0 59.7 65.4 48.0 51.4 48.5 65.6 43.1 41.8 48.8
43.7 60.9 72.1 45.9 51.5 61.0 55.7 60.5 46.5 32.9
78.3 41.7 52.1 49.7 50.6 56.6 54.9 56.2 51.4 62.8

(a) Find a 90% confidence interval for $\mu$, and hence test $H_0: \mu = 50$ against a two-sided alternative, using a test of size 0.10.
(b) Find a 90% confidence interval for $\sigma$, and hence test $H_0: \sigma = 10$ against a two-sided alternative, using a test of size 0.10.

10. Suppose that $X \sim N(\theta, \sigma^2)$. Then, for a random sample of $n = 25$ observations on $X$, the sample mean $\bar{X} \sim N(\theta, \sigma^2/n)$. Show that to test the hypothesis $H_0: \theta = 50$ against the alternative hypothesis $H_1: \theta > 50$ using a test of size $\alpha = 0.05$, we should reject $H_0$ if $\bar{X} > 51.6449$.

Count the number of times that $H_0$ is rejected in each case. For example, for $\theta = 50$, the following commands are used:

MTB > RANDOM 1000 C1;
SUBC> NORMAL 50 1.
MTB > LET C2 = SIGN(C1 - 51.6449)
MTB > TALLY C2
Hence plot an estimate of the power function for $50 \leq \theta \leq 55$.

11. Simulate problem 4.3 using Minitab:

If $X_1 \sim N(16, 2.5^2)$ then 1000 observations on $X_1$ are put into column C1 by the commands:

MTB > RANDOM 1000 C1;
SUBC> NORMAL 16 2.5.

This is supposed to be the blood counts for 1000 healthy people. You reject $H_0$ (healthy) for the lower 10% of this distribution — so that 10% of healthy individuals are incorrectly classed as anaemic. Find the 10% quantile of your sample distribution. It should be somewhere near the theoretical value of 12.8.

Now produce 1000 observations on $X_2 \sim N(9, 3^2)$ and determine how many anaemic individuals are classed as anaemic. This is the power of the test.

Draw dotplots of the two distributions on the same scale and indicate your critical value on the dotplots.

12. [cf. HW=4.3] The likelihood function for an observed set of data is given by:

$$L(\theta) = e^{-400\theta} \theta^{10} (1 + \theta)^{12} \quad (\theta > 0).$$

Test the hypothesis $H_0: \theta = 0.6$ vs $H_1$: $\theta < 0.6$ using each of the following:

1. $H_0 \Rightarrow \hat{T} \sim N(\theta_0, 1/I(\theta_0))$ [MLE-based test];
2. $H_0 \Rightarrow D_1(\theta_0) \sim N(0, I(\theta_0))$ [score test];
3. $H_0 \Rightarrow \ln L(\hat{T}) - \ln L(\theta_0) \sim \chi^2_1$ [ELRT].

13. If $X \sim \Gamma(\theta)$ derive the form of the likelihood ratio test, based on a sample of $n$ observations, to test $\theta = \theta_0$ vs $\theta > \theta_0$.

14. If $X \sim \exp(\theta)$ derive the form of the score test, based on a sample of $n$ observations, to test $\theta = 1$ vs $\theta > 1$.

15. A random sample of $n$ observations is obtained on $X$ which has the Pareto distribution, which has pdf:

$$f(x) = \frac{\theta}{c} \left(\frac{c}{x}\right)^{\theta+1} \quad (x > c)$$

Determine the form of the likelihood ratio test of $H_0: \theta = 1$ vs $H_1: \theta = 0.5$. It is assumed that $c$ is known.

If $n = 20$, specify precisely the likelihood ratio test of size 0.05.