Revison Exercises 1

1. (a) $X$ is a random variable (an estimator), $\bar{x}$ its realisation (an estimate); $\mu$ is a parameter; $\text{E}(X) = \mu$.
   (b) the standard deviation, $\text{sd}(X) = \sqrt{\text{var}(X)}$ is a measure of spread for any random variable, $X$; the standard error, $\text{se}(T)$ is an estimate of the standard deviation of an estimator: $\text{se}(T) = \text{sd}(T)$.

2. $P \approx 0.021$, using a normal approximation; reject $H_0$ — there is significant evidence that the coin is not fair.

3. $\Pr(X_1 \leq x) = \prod_{i=1}^{n} \Pr(X_i \leq x) = (\frac{1}{2})^n; \ (\frac{1}{2})^n = 0.05 \Rightarrow c = 0.05^{1/n}\theta; \ 17.26.$

4. $\hat{\theta} = \frac{n}{T} \cdot \text{var}(\hat{\theta}) = \frac{(\frac{n}{T})^2n^2}{T^2}; \ \hat{\theta} = 3.105, \text{se}(\hat{\theta}) = 0.99.$

5. (a) i. the realisation of a random interval which contains the parameter $\theta$ with probability 0.95;
   ii. the realisation of a random interval which contains the random variable $X$ with probability 0.95.
   (b) i. $14.34, 15.12$; ii. $12.73, 16.73$.

6. (a) 0.0668; (b) 0.8556; (c) 0.9228.

7. $\frac{\partial \ln L}{\partial \mu} = -3 + \frac{36}{\mu} \Rightarrow \hat{\mu} = 11; \ \frac{\partial^2 \ln L}{\partial \mu^2} = -\frac{36}{\mu^2} \Rightarrow \text{se}(\hat{\mu}) = 2.$

8. (a) Histogram with frequency represented by area, thus ordinates are (proportional to) 8 (0 < $x < 1$);
   interpolation gives $m \approx 10.3$.
   (b) $\hat{\theta} = 0.38; \ F(20) = 0.72$; linear interpolation gives $\hat{\mu} \approx 36.9$.
   (c) $\bar{x} \approx \frac{1}{n} \sum f_i u_i = 21.8$.
   (d) $(1 - e^{-\theta})^8(e^{-\theta} - e^{-50})^{11}(e^{-50} - e^{-200})^{17}(e^{-200} - e^{-100})^{14}.$

9. reject $H_0$ if $\Sigma x^2 < k$.

10. (a) $\text{co}_5 = \theta; \ \frac{x^2}{\text{n}(\text{co}_5)} = \frac{0.25}{(\frac{5}{1})^2} = \frac{4\theta^2}{n} \Rightarrow \frac{\theta - 0.92}{\sqrt{\theta}} < \frac{\text{co}_5}{\theta} < \frac{\theta + 0.92}{\sqrt{\theta}} \Rightarrow \ 6.47 < \theta < 37.27.$ (b) $\mu = \infty$.

11. (a) frequency polygons; (b) $\chi^2 = \sum (2 - \text{o})^2 = 6.35 < \chi_0.95^2(3) = 7.815,$ so we accept $H_0$;
   (c) $\bar{m} = 1.56, \bar{s}_m^2 = 0.8752; \bar{n} = 1.46, \bar{s}_n^2 = 1.0186 \Rightarrow (-0.18, 0.38), \text{using est } \pm 2\text{se}.

12. $0.317 < \sigma < 0.619$, using $\frac{1.96^2}{\sigma^2} = \chi^2_{15}; \text{normal plot, QQ plot}.$

13. (a) $\bar{x}_1 = 36.14, s_1 = 6.012; \bar{x}_1 = 37.43, s_1 = 5.623; t = 0.413$, so accept $H_0$;
   (b) $w_1 = 1 + 3 + 11 + 7 + 9\frac{1}{2} + 12\frac{1}{2} + 5 = 49$ (and $w_2 = 6 + 4 + 8 + 14 + 12\frac{1}{2} + 9\frac{1}{2} + 2 = 56$);
   since $37 \leq w \leq 68$, we accept $H_0$.

14. (a) $\Pr(31 \leq Z \leq 49) \approx \Pr(30.5 \leq \bar{Z} < 49.5), \text{where } \bar{Z} \sim N(40, 20); \ (x_{(31)}, x_{(50)}) = (37.6, 42.4)$.
   (b) $Z_0 \sim \text{Bi}(n, \theta)$, and $\Pr(\theta \leq \text{freq}(X \leq c) \leq b) = \Pr(\text{co}_5 < c < X_{(k+1)})$.

15. $H_C: \text{E}(Y_{(1)}) = \mu_1 \quad \text{and} \quad H_C^*: \text{E}(Y_{(1)}) = \alpha + \beta x_1, \quad \text{and} \quad \text{E}(Y_{(5)}) = \alpha + \beta x_1, \quad \text{and} \quad \text{E}(Y_{(1)}) = \alpha + \beta x_1, \quad \text{and} \quad \text{E}(Y_{(5)}) = \alpha + \beta x_1$. 

16. (a) (b) $\hat{\alpha} = \frac{15}{3}, \hat{\beta} = \frac{15}{6}; \ \hat{\alpha} + \hat{\beta} = 8, \ \hat{\alpha} + \hat{\beta} = 8$.

17. (b) blocks (3) 123.19 41.06
treatments (3) 68.19 22.73 17.70
effect error total (9) 11.56 1.28 292.94
i. $F_1 = 17.70 > 3.86$, so there is a significant difference between the treatments;
ii. $15 - 5 \pm 2.262 \sqrt{\frac{1.28}{4} + \frac{1}{4}} = (3.19, 6.81)$.  

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Revision Exercises 2

1. (a) i. a 95% confidence interval for a parameter θ: the realisation of a random interval which has probability 0.95 of containing θ.
   ii. a 95% prediction interval for a random variable X: the realisation of a random interval which has probability 0.95 of containing (a future observation on X).
(b) i. a 95% confidence interval for μ: 14.73 ± 2.064 × 2.81/√10 = (13.57, 15.89).
   ii. a 95% prediction interval for X: 14.73 ± 2.064 × 2.81√(1 + 1/10) = (8.82, 20.64).
   iii. using 24S²/σ² ∼ χ²₄, Pr(12.40 < 24S²/σ² < 39.36) = 0.95, and so a 95% confidence interval for σ: (2.19, 3.91).

2. (a) Both ˆM and ˆX are unbiased. var( ˆX) = σ²/3, since σ² = σ²/3 from the summary notes. var( ˆM) ≈ 1/4σ², so that fX(θ) = 1/4.
   So ˆX is (slightly) more efficient.
   (b) p(x | θ) = θ(1 − θ)x = ln p = ln θ + x ln(1 − θ) ⇒ ∂ln p/∂θ = 1/θ − x/(1 − θ) ⇒ ∂²ln p/∂θ² = −1/θ² − x/(1 − θ)²
   τ(θ) = (1 − θ)/θ² ⇒ τ(θ) = (θ − 2)/θ³.
   Therefore the MVB for τ(θ) is given by (2 − θ)²/θn = (θ − 2)(2 − θ)².
   (c) ˆT₂ is slightly biased but more efficient — and the bias is correctable, so ˆT₂ is preferred.

3. (a) L > 0 ⇒ 0 < θ < 1.
   (b) lnL = −10θ + 6 ln θ + 3 ln(1 − θ)
   ⇒ ∂lnL/∂θ = −10 + 6/θ − 3/(1 − θ) ⇒ 0 ⇒ θ = 0.4
   (c) ⇒ ∂²lnL/∂θ² = −6/θ² − 3/(1 − θ)² ⇒ I(θ) ≈ 6/0.4² + 3/0.6² = 45.83
   ⇒ se(θ) ≈ 1/√I(θ) = 0.148.

4. (a) i. μ = 1/θ ⇒ ˆθ = 2x
   ii. L(θ) = 1/θn, provided x₁, x₂, . . . , xₙ < θ; and zero otherwise.
   Therefore L(θ) = 0 if θ < x(n).
   It follows (from a sketch graph or by observing that 1/θn is a decreasing function) that L is maximised at the end-point: ˆθ = x(n).
   (b) Pr(Xi ≤ x) = Pr(X₁. . . . Xₙ ≤ x) = ∏nPr(Xi ≤ x) = (θ/2)n (0 < x < θ)
   (c) cₙ(X(n)) is such that (cₙ/θ)n = q ⇒ cₙ = q¹/nθ. Therefore
   Pr(0.025¹/n < X(n) < 0.975¹/n) = 0.95
   and, with n = 10 and x₁(10) = 2.59, we obtain 0.6915θ < 2.59 < 0.9079θ, so that the required 95% confidence interval is (2.597, 3.745).

5. Independent samples — so either an independent samples t-test or a rank sum test could be used. The dotplot is suggestive of longer tails, so perhaps the rank test might be favoured

The summary statistics for the two samples are as follows:

brand X: n = 10; ˆx = 75.0; sx = 32.5
brand Y: n = 10; ˆy = 58.5; sy = 36.1

Thus there is little difference in the sample variances and so it will matter little whether the pooled or unpooled variance estimate is used.

\[ t = \frac{\overline{x} - \overline{y}}{s_p \sqrt{\frac{1}{n} + \frac{1}{n}}} = 1.08; \] and under H₀, \( t \sim t_{18} \), so we accept H₀. There is no evidence that brand X failures are greater than those for brand Y.

For the rank sum test, the ranks are:

| brand X | 7 | 14 | 8 | 12.5 | 2 | 17 | 15 | 18 | 11 | 19 |
| brand Y | 1 | 4.5 | 16 | 4.5 | 10 | 20 | 3 | 9 | 12.5 | 6 |

so that \( W_X = 123.5 \) (and \( W_Y = 86.5 \)). Therefore we accept H₀ — as for the t-test.
6. (a) $E(X) = 5\theta \Rightarrow \bar{X} = \frac{1}{n} \sum X \Rightarrow \bar{X} = 0.64$, so $\theta = 0.128$.

$$L(\theta) = (1 - 3\theta)^{60} \theta^{16} (2\theta)^{24} \ln L = k + 60 \ln(1 - 3\theta) + 40 \ln \theta \Rightarrow \frac{\partial \ln L}{\partial \theta} = -\frac{180}{1 - 3\theta} + \frac{40}{\theta} = 0 \Rightarrow \hat{\theta} = 0.133.$$

(b) $E(X^2) = 9\theta \Rightarrow \text{var}(X) = 9\theta - 25\theta^2$, so $\text{var}(\theta) = \frac{9\theta - 25\theta^2}{25n} \Rightarrow \text{se}(\theta) = 0.0172$.

$$\frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{540}{(1 - 3\theta)^2} - \frac{40}{\theta^2} \Rightarrow t(\hat{\theta}) \approx 3750 \Rightarrow \text{se}(\theta) = 0.0163.$$

(c) $\hat{\theta} = 1.96 \text{se}(\hat{\theta}) = (0.101, 0.165)$.

(d) Using $\hat{\theta} = 0.1333$, we obtain

So $\chi^2 = \sum \frac{(a - e)^2}{e} = 0.8$ and $df = 1$; thus we accept the goodness of fit of the model.

7. (a) $\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} [\beta] + \varepsilon$.

$A'A\hat{\theta} = A'y \Rightarrow \Sigma x_i^2 [\hat{\beta}] = [\Sigma x_i y_i] \Rightarrow \beta = \frac{\Sigma x_i y_i}{\Sigma x_i^2}$.

$E(\hat{\beta}) = \frac{1}{\Sigma x_i^2} \sum x_i \beta x_i = \beta$.

$\text{var}(\hat{\beta}) = \frac{1}{(\Sigma x_i^2)^2} \sum x_i^2 \sigma^2 = \frac{\sigma^2}{\Sigma x_i^2}$.

(b) $n = 10$, $\Sigma x^2 = 110$, $\Sigma xy = 1175$, $\Sigma y^2 = 12681$.

i. $\hat{\beta} = \frac{1175}{110} = 10.68$

ii. $H_0: E(Y_{ij}) = \beta x_i \Rightarrow SS_0 = 12551.14$ (using $\hat{\theta}'A'y = \frac{[\Sigma x_i y_i]^2}{\Sigma x_i^2}$) with $df = 1$; and hence $d^2_0 = 129.86$ with $df = 9$.

$H_1: E(Y_{ij}) = \mu_i \Rightarrow SS_1 = 12633.5$ (using $21^2 \pm \cdots + 11^2$) with $df = 5$; and hence $d^2_1 = 47.50$ with $df = 5$.

Thus, to test $H_0$, $F = \frac{82.36/4}{47.50/5} = 2.17$, which is not significant (compared to $F_{4,5}$). And so we accept the goodness of fit of the regression model.

iii. $\hat{\beta} = 10.68$, se($\hat{\beta}$) = $\sqrt{\frac{\sigma^2}{\Sigma x_i^2}} = 0.3622$; so a 95% CI for $\beta$ is given by $10.68 \pm 2.626 \times 0.3622 = (9.86, 11.50)$.

8. (a) The simplest method to obtain the sums of squares is probably to find $s_y$ for the given data using a calculator: this gives $s_y = 1.662487$; then the total sum of squares is $11s_y^2 = 30.4025$.

<table>
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<th>MS</th>
<th>F</th>
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<td>8.30</td>
<td>5.41</td>
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<tr>
<td>total</td>
<td>11</td>
<td>30.40</td>
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(b) $F = 5.41 > c_{0.05}(F_{2,9})$ and so we reject $H_0$. $P \approx 0.03$.

(c) $s^2 = \text{error MS} = 1.53$

(d) $\bar{y}_2 - \bar{y}_1 = 30.45 - 28.55 = 1.90$; se($\bar{y}_2 - \bar{y}_1$) = $\sqrt{s^2(\frac{1}{4} + \frac{1}{4})} = 0.875$. So a 95% CI for $\mu_2 - \mu_1$ is given by $1.90 \pm 2.626 \times 0.875 = (-0.08, 3.88)$. 

9. (a) $\begin{bmatrix} 2.1 & 10.4 \\ 4.8 & 8.8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \varepsilon$

(b) $\begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \begin{bmatrix} 42.5 \\ 34.9 \end{bmatrix}$

(c) $\begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \frac{1}{16} \begin{bmatrix} 8 & -4 \\ -4 & 8 \end{bmatrix} \begin{bmatrix} 42.5 \\ 34.9 \end{bmatrix} = \begin{bmatrix} 4.175 \\ 2.275 \end{bmatrix}$

(d) $s^2 = \frac{1}{n-2} (y' y - \hat{\theta}'A'y) = \frac{1}{16} (257 - 256.835) = 0.04125$

(e) $\hat{\alpha} + \hat{\beta} = 6.45$; se($\hat{\alpha} + \hat{\beta}$) = $\sqrt{(\frac{8}{16} - 2 \times \frac{4}{16} + \frac{8}{16})s^2} = 0.082916$; 95% CI for $\alpha + \beta$: 6.45 $\pm$ 2.776 $\times$ 0.0829 = (6.22, 6.68).
10. ANALYSIS OF VARIANCE

We consider three models:

(a) \( F = 10.09 \) and so the treatment effects are significant, as \( c_{0.95}(F_{3,15}) = 3.29 \). We therefore reject the hypothesis that there is no difference between the treatments.

We are assuming an additive model with independent normally distributed errors with equal variances. There is nothing striking in the data to suggest these assumptions are unreasonable.

(b) \( s^2 = 2.13 \); \( Pr(6.26 < 15s^2/\sigma^2 < 27.49) = 0.95 \) \( \Rightarrow \) 95% CI for \( \sigma^2 \) : (1.16, 5.11).

\( \hat{y}_0 - \hat{y}_1 = 0.67 \) and se(\( \hat{y}_0 - \hat{y}_1 \)) = \( \sqrt{s^2(\frac{1}{n_0} + \frac{1}{n_1})} = 0.8433 \). Therefore a 95% CI for \( \tau_2 - \tau_1 \) is 0.667 \( \pm \) 2.131 \( \times \) 0.843 = (\(-1.13, 2.46\)). And similarly for the other differences.

11. (a) \( r = 0.927 \) = 0.960.

(b) We consider three models:

Straight line (L): \( E(Y_{ij}) = \alpha + \beta x_i \)

Quadratic (Q): \( E(Y_{ij}) = \alpha + \beta x_i + \gamma x_i^2 \)

General (G): \( E(Y_{ij}) = \mu_i \ [= g(x_i)] \).

From the given output we have the split-ups of the sum of squares (corrected for the mean) with each of the models:

\[
\begin{array}{cccccc}
\text{model} & \text{Straight line} & \text{Quadratic} & \text{General} \\
\hline
\text{residual} & 39.65 & (8) & 9.61 & (7) & 5.50 & (5) \\
\end{array}
\]

Testing the goodness of fit of the Straight line model has \( H_0 = L \) and \( H_1 = G \), and \( F = 4.15/5 = 10.34 \), which is greater than \( c_{0.95}(F_{3,5}) = 5.41 \). So we reject the goodness of fit of the straight line model.

Testing the goodness of fit of the Quadratic model has \( H_0 = Q \) and \( H_1 = G \), and \( F = 4.11/2 = 5.00/5 \) = 1.87, which is less than \( c_{0.95}(F_{2,5}) = 5.79 \). So we accept the goodness of fit of the quadratic model.

(c) Let \( \eta = E(Y | x = 3) \).

i. Assuming a straight line regression model, \( \hat{\eta} = \hat{\alpha} + 3\hat{\beta} = \bar{y} + \hat{\beta} = 26.7 \); and se(\( \hat{\eta} \)) = \( \sqrt{\frac{3.09}{15}} + 0.4978^2 = 0.862 \), so a 95% CI for \( \eta \) = 26.7 \( \pm \) 2.306 \( \times \) 0.862 = (24.71, 28.69).

ii. Making no assumptions about the regression, \( \hat{\eta}_2 = 32.5 \); and se(\( \hat{\eta}_2 \)) = \( \sqrt{\frac{1.10}{2}} = 0.742 \), so a 95% CI for \( \eta \) = 32.5 \( \pm \) 2.306 \( \times \) 0.742 = (30.59, 34.41).

Revision Exercises 3

1. (a) 19.56, 20.44; \( b \) \( Z \frac{\hat{d}}{\hat{d}} \) Bi(500, 0.0907), (33, 58); \( c \) \( n \approx N(10, \sigma^2/400) \), \( d \) 1 \( (1 - e^{-5})^{80} \) = 0.418; \( e \) \( Pr(x_{\hat{d}}^2 > 50.68) \approx 0.10 \).

2. \( x = 28.13, s = 4.04 \); (a)i. 28.13; (b) \( 24.26, 34.02 \); ii. accept \( \mu = 30 \); (b)ii. 4.04; (3.07, 5.90); ii. accept \( \sigma = 5 \); (c) (19.46, 36.80).

3. (a) \( L(\theta) = \theta^{66}(1 - \theta)^{95} \Rightarrow \hat{\theta} = 0.41, se(\hat{\theta}) = 0.039 \); \( b \) exp.freq: 32.8, 19.4, 11.4, 16.4; \( \chi^2 = 2.8 \), so accept goodness of fit.

4. (a) \( \mu = e^{\theta + \frac{1}{2}} \Rightarrow \hat{\theta} = (\ln \bar{y} - \frac{1}{2}); \) \( var(\hat{\theta}) = \frac{1}{n} e^{2\theta + 1}(e - 1) \Rightarrow var(\hat{\theta}) \approx \frac{e^\theta}{n} \); \( b \) \( \hat{\theta} = \bar{x}, \) where \( x = \ln \bar{y}, \) \( var(\hat{\theta}) = \frac{1}{n} \); \( c \) both consistent; \( \hat{\theta} \) biased, \( \hat{\theta} \) unbiased; \( \hat{\theta} \) is more efficient since \( var(\hat{\theta}) < var(\theta) \).

5. (a) \( L(\alpha, \beta) = (1 - \alpha)^{16}\alpha^3(1 - \beta)^5 \Rightarrow \hat{\alpha} = 0.2, se(\hat{\alpha}) = 0.089; \hat{\beta} = 0.44, se(\hat{\beta}) = 0.166 \).

(b) \( \hat{\beta} - 2\alpha = 0.044, se(\hat{\beta} - 2\alpha) = 0.243; x = 0.18, \) so accept \( \beta = 2\alpha \).

6. (a) \( P = Pr(X > 12.3 | H_0) = Pr(X > 3.637) = 0.000 \); \( b \) \( L(\theta) = \theta^{20} e^{-\theta x_i} \Rightarrow reject \ H_0 \ if \ \Sigma x_i > k; \) 2602 \( \Sigma x_i \approx \chi^2 \Rightarrow k = 55.76; \) power = \( Pr(\chi^2 > 22.30) = 0.99 \).

7. (a) dotplots; \( b \) \( \bar{x}_2 - \bar{x}_1 = 25.83; se(\bar{x}_2 - \bar{x}_1) = 12.27; \) \( se(\bar{x}_2 - \bar{x}_1) = 9.95 \); \( t_2 = 1.70, t_p = 2.09; \) \( c_{0.975}(ts) = 2.306 \). \( c \) \( W_2 = 6 + 8 + 9 + 10 = 33 \); tables \( \Rightarrow reject \ H_0 \ unless \ 13 \leq W_2 \leq 31, \) so reject \( H_0 \).
(d) outlier in sample 2 inflates the variance estimate which reduces $t$, but its effect on the rank test is limited.

8. (a) $\hat{m} = 142$, ii. $Z = \text{freq}(X \leq m) \overset{d}{=} \text{Bi}(80, \frac{1}{2}) \Rightarrow \Pr(Z \leq 49) \approx 0.967$; 96.7% CI: $(x_{(31)}, x_{(50)}) = (124, 187)$. (b) i. QQ-plot ii. using $\hat{x}$, obtain a CI for $m = m / \ln 2$ and hence a CI for $m$.

9. (a) $F = \text{devMS/resMS} = (0.95/3)/(19.50/5) = 0.08 \Rightarrow$ excellent fit.

(b) $\hat{\eta} = 0.85$, se($\hat{\eta}$) = $\sqrt{\frac{2.56}{10}} + 9 \times 0.375^2 = 1.186$, 95% CI: 0.85 ± 2.306 × 1.186 = (−1.88, 3.58).

10. (a) $\begin{bmatrix} 5 \\ 8 \\ 6 \\ 10 \\ 12 \\ 15 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} + \varepsilon$; $\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{\gamma} \end{bmatrix} = \begin{bmatrix} 30 \\ 45 \\ 33 \end{bmatrix}$; $\begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{\gamma} \end{bmatrix} = \begin{bmatrix} 3.75 \\ 6.75 \\ 5.25 \end{bmatrix}$

$s^2 = \frac{1}{5}(594 - 589.5) = 1.5$. (b) $\hat{\eta} = 15.75$, se($\hat{\eta}$) = $\sqrt{\frac{1}{5} s^2} = 0.866$; 95% CI: 15.75 ± 3.182 × 0.866 = (12.99, 18.51). (c) $t = \frac{\hat{\eta} - \eta}{\text{se}(\hat{\eta})} = \frac{15.75 - 15}{0.866} = 1$, so accept $QR = RS$; (d) $A' A = 20$, $A'y = 108$, $\hat{\theta} = 5.4$, $SS = 583.2$ (df = 1); $F_{2,3} = \frac{\text{devMS}}{\text{resMS}} = \frac{3.15}{1.5} = 2.1$, so accept $PQ = QR = RS$.

11. (a) $\begin{array}{ccc} B & (5) & 367.97 \\ T & (2) & 367.72 \\ E & (28) & 298.20 \\ \Sigma & (35) & 1033.20 \end{array}$ $F_{2,28} = \frac{183.86}{10.65} = 17.26$, so treatment effects are significant.

(b) $\bar{x}_p - \bar{x}_c = 23.25 - 15.50 = 7.75$, se($\bar{x}_p - \bar{x}_c$) = $\sqrt{10.65(\frac{1}{72} + \frac{1}{72})}$; 95% CI: 7.75 ± 2048 × 1.33 = (5.02, 10.48).

Revision Exercises 4

1. (a, b)

(c) $\bar{x} = \frac{18350}{200} = 66.75$; $s^2 = \frac{1}{199}(947750 - \frac{18350^2}{200}) = 284.61$; $s = 16.87$.

(d) $66.75 \pm 1.972 \times \frac{16.87}{\sqrt{200}} = (44.4, 69.1)$.

(e) $z = \text{freq}(X < 70) = 120$; $z_a = \frac{119.5 - 100}{\sqrt{50}} = 2.76$; so reject $H_0$.

2. (a) $\Pr(a(\lambda) \leq X \leq b(\lambda)) \geq 0.95$; $a(\lambda)$ is the largest value such that $\Pr(X \geq a(\lambda)) \geq 0.975$, and $b(\lambda)$ the smallest value such that $\Pr(X \leq b(\lambda)) \geq 0.975$.

(b) i. 11 ≤ X ≤ 29; ii. From the SP diagram, 15 < 20λ < 36, and so 95% CI: 0.75 < λ < 1.8.

3. (a) i. $\Pr(\hat{X}_a > \frac{0.4}{\sqrt{13}}) = \Pr(N > 1.62) = 0.123$;

ii. $\Pr(14562 > 14 \times 1.3^2) = \Pr(\hat{X}_a^2 > 23.66) = 0.050$;

iii. $\Pr(\hat{X}_a(15) > 2.3) = 1 - 0.9893^{15} = 0.149$;

iv. $N \overset{d}{=} \text{Bi}(15, \frac{1}{2})$, so $\Pr(N \geq 10) = 0.151$, from Binomial tables.

(b) $N(0, \frac{1}{15} + \frac{1}{13}) = N(0, \frac{1}{14.9})$; $F_{14.9}$: $\chi^2_2$: $\chi^2_{13}$.

4. (a) $\ln L = n \ln \theta - (\theta + 1) \sum \ln (1 + x_i)$; $\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} - \sum \ln (1 + x_i) \Rightarrow \hat{\theta} = n/(\sum \ln (1 + x_i))$; $\frac{\partial^2 \ln L}{\partial \theta^2} = -n/\theta^2 \Rightarrow \text{se}(\hat{\theta}) = \hat{\theta}/\sqrt{n}$.
(b) \(1 - F = (1 + x)^{-\theta} \Rightarrow -\ln(1 - F) = \theta \ln(1 + x)\); since \(F'(x(k)) \approx \frac{1}{n}\), we should have \(-\ln(1 - \frac{k}{n}) \approx \theta \ln(1 + x(k))\) [QQ plot].

If the model is correct, the plot should be close to a straight line through the origin with slope \(\theta\).

It gives a model check and allows outlier detection and is thus fairly robust, but may be less efficient if the model is correct.

(c) \(Pr(Y > y) = Pr(\ln(1 + X) > y) = Pr(X > e^y - 1) = e^{-\theta y}, \quad (y > 0)\);

hence \(Y \sim \exp(\theta)\). It follows that \(\Sigma Y \sim \Gamma(10, \theta)\), and hence \(29\Sigma Y \sim \chi_{19}^2\).

Therefore \(Pr(9.591 < 29\Sigma Y < 34.17) = 0.95 \Rightarrow 95\% \text{ CI}: 0.68 < \theta < 2.43\).

5. i. unbiased \(\Rightarrow a_1 + a_2 = 1;\) efficient \(\Rightarrow \min V = a^2\sigma^2 + (1 - a)^2\sigma^2_2;\) \(\frac{\partial V}{\partial a} \Rightarrow a = \frac{\sigma^2_1}{\sigma^2_1 + \sigma^2_2}\).

ii. \(a = 0.8177; \quad \hat{\theta} = 2.53, \text{se}(\hat{\theta}) = 0.31\).

iii. weights inversely proportional to variance: \(w_i = \frac{1}{\text{se}(\hat{\theta}_i)^2}\).

6.(a) \(\hat{m} = x_{(34)} = 1.58; \quad \hat{c}_{0.25} = x_{(17)} = 0.84, \quad \hat{c}_{0.75} = x_{(51)} = 2.73\).

(b) \(Z^* \sim N(33.5, 16.75)\); \(Pr(26 \leq Z \leq 41) \approx Pr(25.5 < Z^* < 41.5) = Pr(-1.955 < Z^* < 1.955) = 0.949\), from tables. 95% CI \(= (x_{(26)}, x_{(42)}) = (1.12, 2.10)\).

(c) \(Pr(Y^* \in (Y_{(2)}, Y_{(60)})) = \frac{64}{84} = 0.941\).

(d) \(\ln Y \sim N(\alpha, \beta^2)\), where \(m = e^{\alpha}\).

i. 95% CI for \(m\): \(0.415 \pm 0.670 \times \frac{0.670}{\sqrt{27}} = (0.252, 0.578)\); and hence 95% CI for \(m\) is given by \((1.29, 1.78)\) [which is much narrower than the CI in (b)]

ii. 95% PI for \(\ln Y\): \(0.415 \pm 0.670 \times 0.670 \times \frac{1}{\sqrt{27}} = (-0.933, 1.763)\); and hence 95% PI for \(Y\) is given by \((0.39, 5.83)\) [which is rather wider than the PI in (c)!!?]

7. i. \(H_0 \Rightarrow T \sim N(4, \frac{1}{5})\), and \(H_1 \Rightarrow T\) larger; so we reject \(H_0\) if \(T > 4 + 1.6449 \times \frac{1}{\sqrt{5}}\), i.e. if \(T > 4.658\).

ii. power = \(Pr(T > 4.658 | \theta = 5)\), and \(\theta = 5 \Rightarrow T \sim N(5, \frac{1}{25})\); so power = \(Pr(T > -0.765) = 0.778\).

iii. size = 0.05 \(\Rightarrow \frac{c-4}{\sqrt{n}} = 1.6449 \Rightarrow c = 4 + 1.6449 \times \frac{1}{\sqrt{n}}\)

power = 0.90 \(\Rightarrow \frac{c-5}{\sqrt{n}} = -1.2816 \Rightarrow c = 5 - 1.2816 \times \frac{1}{\sqrt{n}}\)

And then solve these two equations for \(n\) and \(c\).

8. Paired samples so consider the sample of differences:

\[
\begin{array}{cccccccc}
\text{diff} & -2 & 1 & 7 & -2 & 1 & 3 & 3 & 4 & 5 & -1 \\
\text{rank} & 4 \frac{1}{2} & 2 & 10 & 4 \frac{1}{2} & 2 & 6 \frac{1}{2} & 6 \frac{1}{2} & 8 & 9 & 2
\end{array}
\]

i. \(n = 10, d = 1.9, s_d = 3.035; t = \frac{d}{s_d/\sqrt{n}} = \frac{1.9}{3.035/\sqrt{10}} = 1.980, \text{ cf. } c_{0.975}(t_9) = 2.262, \text{ so accept } H_0\).

ii. \(v_c = 11, v_a = 44 \Rightarrow P > 0.10, \text{ and so we accept } H_0\).

An approximate 95% CI for \(d\) is given by \(1.9 \pm 2.262 \times 0.9597 = (-0.27, 4.07)\).

9. No. There is an obvious outlier in group 3; perhaps the 54 should be 24? In any case it should be investigated and an explanation sought. Either omit it and re-do the ANOVA or use a rank-based test. It is then likely that the differences will be found to be significant.

10.(a)

(b) \(r = 0.8;\)

(c) \(0.14 < \rho < 0.61; \text{ yes, } 0 \notin C;\)

(d) if there are outliers; if the relation is non-linear.
11. (a) $L(\beta) = \prod_{i=1}^{n} \frac{e^{-\beta x_{i}}(\beta x_{i})^{y_{i}}}{y_{i}!}$. \[ \ln L = -\beta \sum x_{i} + \ln \beta \sum y_{i} + k. \]

\[ \frac{\partial \ln L}{\partial \beta} = -\sum x_{i} + \frac{1}{\beta} \sum y_{i} \Rightarrow \beta = \frac{\sum y_{i}}{\sum x_{i}}. \]

\[ \frac{\partial^{2} \ln L}{\partial \beta^{2}} = -\frac{1}{\beta^{2}} \sum y_{i} \Rightarrow I(\beta) = \frac{\sum x_{i}^{2}}{\beta^{2}}; \text{ so } \text{MIV} = \frac{\beta}{\sum x_{i}^{2}}. \]

\[ \text{var}(\beta) = \frac{\sum x_{i}^{2}}{(\sum x_{i})^{2}} = \frac{\beta}{\sum x_{i}^{2}} = \text{MIV}, \text{ so } \beta \text{ is the MIV estimator.} \]

Sample data: $\sum x_{i} = 36.7$, $\sum y_{i} = 45 \Rightarrow \hat{\beta} = 1.23$, \[ \text{se}(\hat{\beta}) = 0.18. \]

(b) $E(Y_{w}^{2}) = K^{-1/2} A_{w} \theta = \theta_{w}$; $D(Y_{w}) = K^{-1/2}(\sigma^{2} K)K^{-1/2} = \sigma^{2} I$

Applying standard least squares to $Y_{w}$ gives the estimator $\hat{\theta}_{w}$ which is such that $A_{w}^{\prime} A_{w} \hat{\theta}_{w} = A_{w}^{\prime} Y_{w}$, i.e., $A^{\prime} K^{-1} A \hat{\theta}_{w} = A^{\prime} K^{-1} Y$.

Since $\hat{\theta}_{w}$ is the LS estimate, $\hat{\theta}_{w}$ is BLUE (best linear unbiased estimator); and therefore must be better than $\hat{\theta}_{w}$.

12. (a) The observational equations and normal equations are given by:

\[
\begin{bmatrix}
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\begin{bmatrix}
5 \\
12 \\
\alpha_{1} \\
\alpha_{2} \\
\alpha_{3} \\
\beta \\
5 \\
30 \\
40 \\
60 \\
80 \\
60
\end{bmatrix}.

Therefore: \[ \begin{bmatrix} \hat{\alpha}_{1} \\ \hat{\alpha}_{2} \\ \hat{\alpha}_{3} \\ \hat{\beta} \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \\ 16 \\ 2 \end{bmatrix}; \text{ and } s^{2} = \frac{1}{15-4}(2448 - 2440) = \frac{8}{11}. \]

(b) 95% CI for $\beta$: \[ 2 \pm 2.201 \times \sqrt{\frac{2}{11}} = (1.66, 2.34). \]

(c) 95% CI for $\gamma = \alpha_{3} + 2\beta$: \[ 20 + 2.201 \times \sqrt{\frac{8}{11}(\frac{1}{5} + \frac{4}{30})} = (18.92, 21.08). \]

13. (a) Analysis of variance:

\begin{center}
\begin{tabular}{|c|c|c|c|}
\hline
Source & df & Sum of Squares & Mean Square \\
\hline
$H_{0}$ & 5 & 31230 & \\
$\text{dev}$ & 3 & 90 & 30 \\
$\gamma y$ & 17 & 136 & 8 \\
Total & 25 & 31546 & \\
\hline
\end{tabular}
\end{center}

\[ F = \frac{\frac{\text{dev MS}}{\text{df}}}{\frac{\gamma y}{\text{MS}}/8} = \frac{30}{8} = 3.75 > c_{0.05}(F_{3,17}) = 3.20, \] so we reject $H_{0}$ (and conclude that the model $H_{0}$ is not a good fit to the data).

(b) i. \[ \text{blocks} : 7, 154, 22 \]

\[ \text{treatments} : 2, 24, 12 \]

\[ \text{error} : 38, 76, 2 \]

\[ \text{total} : 47, 254 \]

\[ F_{\text{treatments}} = 6, \text{ cf. } c_{0.05}(F_{2,38}) = 3.25, \text{ so we reject } H_{0}, \text{ and conclude that the treatment effects are significant.} \]

\[ ii. \frac{385^{2}}{\sigma^{2}} \approx \chi_{3}^{2}. \text{ 95% CI: } 22.88 < \frac{385^{2}}{\sigma^{2}} < 56.90 \Rightarrow 1.34 < \sigma^{2} < 3.32. \]

\[ iii. \text{est} = 2.3, \text{se} = \sqrt{s_{2}(\frac{1}{15} + \frac{1}{30})} = \frac{1}{\sqrt{4}} = 0.5; \text{ 95% CI: } 2.3 \pm 0.204 \times 0.5 = (1.29, 3.31). \]

Revision Exercises 5

1. (a) $\Pr(X = 1) = 0.05$, $Z \overset{d}{=} B_{15}(0.05)$, $\Pr(Z \geq 2) = 0.17$;
(b) $\Pr(W(1) > 1) = \Pr(W > 1)^{10} = 0.5^{10} = 0.0010; \Pr(W(10) < 9) = \Pr(W < 9)^{10} = 0.9^{10} = 0.3487$.

2. (i) $F(c_{0}) = q \Rightarrow c_{0} = \ln \frac{q}{1-q}$; \[ \text{ii. } c_{0}(Y) = \theta + \phi c_{0}(Z), \text{ so } Y_{(k)} \approx \theta + \phi(n+1)k; \text{ close to a straight line if the model is correct and } \theta = \text{ intercept, } \phi = \text{ slope}; \text{ iii. sketch with } \theta \text{ and } \phi; \text{ iv. } t_{0.5}(Y) \overset{d}{=} N(0, \frac{1}{n+1}).]
3. (a) \( E(U_n) = e^\theta \), var\( (U_n) = e^{2\theta} n \to 0 \) as \( n \to \infty \); therefore \( U_n \xrightarrow{p} e^\theta \), and hence \( \ln U_n \xrightarrow{p} \theta \), since \( \ln \) is continuous at \( e^\theta \). (b) \( 1.54 < \lambda < 3.57 \). (c) (i) \( \Pr(\frac{1}{2}\theta < T < \frac{3}{2}\theta) = 0.95 \Rightarrow 95\% \text{CI}: \theta > 45 \); (d) graph (Table 2) \( \theta > 0.13 \) or \( \theta < 0.29 \). (e) \( \frac{135^2}{10} \Rightarrow \Pr(T < \frac{135^2}{25}) = 0.95 \Rightarrow 95\% \text{CI}: \theta = 2.6 > \sigma^2 > 13. \\

4. (a) i. size = 0.0511, power = 0.7798, \( P = 0.0214 \); ii. reject \( \theta = 4 \) \( (P < 0.05) \), accept \( \theta = 5 \) \( (P > 0.05) \). iii. \( g = 1.645 \times 13 = 41 \). (b) \( L(\theta) = \theta e^{-\theta} 25; L_1/L_0 = (\theta_1/\theta_0)^e e^{-(\theta_1 - \theta_0) 25} \Rightarrow \) reject \( H_0 \) if \( y_2 < k; \) UMP test since it is independent of \( \theta_1 \). (c) consistent, since \( E(T_n) \to \theta \), \( \text{var}(T_n) \to 0 \); not unbiased, since \( E(T_n) \neq \theta \); asymptotically unbiased, since \( \text{var}(T_n) \to \theta \); not efficient, since \( \text{var}(T_n) / \text{var}(\tilde{\theta}) = (\theta^2 + 1)/\theta^2 \neq 1 \). (b) \( \tilde{\theta} = \frac{1}{n} (x - \bar{x}) = 5 \), var\( (\tilde{\theta}) = \frac{26^2}{10 n} \Rightarrow \text{se}(\tilde{\theta}) = \frac{26}{\sqrt{10n}} = \frac{5}{12} \). (c) \( \ln \theta = \ln(12) - \ln(36) + \frac{4}{2} \sqrt{2} \theta_1^2 - \frac{4}{2} \theta_2^2 \Rightarrow \) \( n = 2 \theta_1 - 1 \) \( \Rightarrow \text{se}(\theta) = \frac{12}{\sqrt{n}} \); approx 95\% CI: \( 2 \pm 1 = 1, 3 \); \( \tilde{\theta} = \frac{\ln(\theta_1)}{\ln(\theta_2)} \) as \( n \to \infty \). (d) \( \frac{25}{\sigma^2} \frac{\sigma^2}{5} = \chi^2_7 \Rightarrow \frac{0.2 \sqrt{2}}{2} < \frac{1}{2} < \frac{5 \sqrt{2}}{2} \Rightarrow 0.5 < \frac{25}{\sigma^2} < 2.5 \); \( 1 \in \text{CI} \), so accept \( \sigma_1 = \sigma_2 \); ii. \( s^2 = 36; \chi^2 = \frac{143^2}{5} = \chi^2_{14} \). (iii. \( \bar{x} = 30 \pm 2 \frac{45}{2} \frac{2}{3} (23.6, 36.4); t = \frac{30 - 3}{3} = 10 \); iv. \( df = (1, 14), F = l^2 = 100; \) WMS = \( s^2 = 36. \\

5. i. df SS MS F P II. \( \tilde{x}_i, \tilde{x}_i, \tilde{x}_i \text{ and CI for } \mu_i; \) CI for \( \mu_i - \mu_j \); family error rate. 3 60 20 5 0.008 36 144 4 39 204  \\

6. i. boxplot \[ \text{min} = 124, \text{Q1} = 318.75, \text{med} = 389, \text{Q3} = 465.75, \text{max} = 759 \] ii. \( Z = \frac{200}{10^2} \approx N(200, 10^2) \); so \( \Pr(180 < Z < 220) \approx \Pr(-2.05 < Z < 2.05) \approx 0.96; \) CI = \( (373, 405) \); iii. \( \bar{x} \approx 390, s \approx 105; \) CI = \( 390 \pm 2 \times \frac{105}{\sqrt{105}} \approx (380, 400), \) narrower. iv. PI \( \approx (190, 610) \). 9. The sign test on the signed rank sum test indicate acceptance of \( H_0 \) while the t-test indicates rejection of \( H_0 \). The sign test makes no assumptions but is less-powered; and thus unlikely to reject \( H_0 \) even when it is false. The SRS test assumes symmetry and the t-test assumes normality of the underlying population. If these assumptions are true (or nearly true) then these tests have greater power. If the data are normal (or symmetric) then the mean and the median are the same. The sample is too small to make any judgement on normality (or symmetry). And if the population is slightly positively skew then the median will be less than the mean. The only question that arises concerns the largest observation, but the only effect of this observation is to make it less likely that the t-test would reject \( H_0 \): the data are indicating the median/mean/middle is less than 30. The large observation makes the SRS test less likely to reject \( H_0 \) too. All three confidence intervals are indicating that 30 is around the upper limit of values compatible with the data.  "Assuming the underlying population is roughly normal, there is significant evidence in the sample against the hypothesis that the median is 30".  

10. (a) \[ \begin{bmatrix} y_1 \\
    y_2 \\
    y_3 \\
    y_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\
    0 & 1 \\
    1 & 1 \\
    1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\
    \beta \\
    \xi \end{bmatrix} = \begin{bmatrix} \hat{\alpha} \\
    \hat{\beta} \\
    \hat{\xi} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 0 \\
    0 & 1 \\
    1 & -1 \end{bmatrix} \begin{bmatrix} y_1 + y_3 \\\n    y_2 + y_4 \end{bmatrix}. \]

\( E(\hat{\Lambda}) = \frac{1}{3}(\alpha + \beta + \alpha - \beta) = \alpha \); var\( (\hat{\Lambda}) = \frac{1}{3}\sigma^2 + \sigma^2 + \sigma^2 = \sigma^2 \). (b) LS is BLUE = the least squares estimator is the best linear unbiased estimator. (c) \( H_0 \) \( F = \frac{32}{20} = 4 \); cf. \( c_0.95(F_{4,20}) = 2.866, \) so reject \( H_0 \).  

11. (a) derivation of \( A^T A \hat{\theta} = A^T \hat{\theta} \) (see notes): \[ \begin{bmatrix} \hat{\alpha} \\
    \hat{\beta} \\
    \hat{\xi} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\
    0 & 1 & 0 \\
    1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 30 \\
    16 \\
    6 \end{bmatrix} = \begin{bmatrix} 3 \\
    2 \\
    1 \end{bmatrix}; \]

\( \hat{\theta}^T A^T \hat{\theta} = 90 + 32 + 6 = 128; \) ii. \( s^2 = \frac{1}{15} (158 - 128) = 2.5; \) iii. se\( (\tilde{\theta}) = \frac{1}{20} = 0.5; \)

se\( (\hat{\alpha} + \hat{\beta}) = \sqrt{\frac{2}{15} - \frac{4}{15} + \frac{2}{15}} = \sqrt{\frac{2}{15}} = 1.1. \)

12. (a) random allocation; two \( T_i \) in each block; random ordering of 11223344. (b) \( B \) \( (4) \) 1200 300 15 treatments significant since \( F = 10 > c_{0.95}(F_{4,32}) = 2.9. \)

\( T \) \( (3) \) 600 200 10 \( E \) \( (32) \) 640 20 \( \Sigma \) \( (39) \) 2440 \( \hat{y}_1 - \hat{y}_2 \pm \sqrt{\frac{2}{15} (158 - 128)} = 10 \pm 2.037 \sqrt{\frac{20}{15} + \frac{1}{15} = (5.9, 14.1). \)