### Comparison of two normal populations

<table>
<thead>
<tr>
<th>population</th>
<th>sample</th>
<th>$X_1$</th>
<th>$X_{11}, X_{12}, \ldots, X_{1n_1}$</th>
<th>$\bar{X}_1$</th>
<th>$S_1^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1 \sim N(\mu_1, \sigma_1^2)$</td>
<td>$X_2 \sim N(\mu_2, \sigma_2^2)$</td>
<td>$X_1, X_{21}, X_{22}, \ldots, X_{2n_2}$</td>
<td>$\bar{X}_2$</td>
<td>$S_2^2$</td>
<td></td>
</tr>
</tbody>
</table>

We wish to make comparisons between $\mu_1$ and $\mu_2$; and (less often) between $\sigma_1^2$ and $\sigma_2^2$.

**F-distribution** (Notes 202=p83)

**Definition**: If $U_1 \overset{d}{=} \chi^2_{\nu_1}$, $U_2 \overset{d}{=} \chi^2_{\nu_2}$ and $U_1 \ & U_2$ are independent, then $Z = \frac{U_1/\nu_1}{U_2/\nu_2} \overset{d}{=} F_{\nu_1,\nu_2}$

(F-distribution with $\nu_1$ and $\nu_2$ degrees of freedom)

![pdf of F-distribution](image)

Note: [1] $E(F_{\nu_1,\nu_2}) = \frac{\nu_2}{\nu_2 - 2}$; [2] $\frac{1}{Z} = \frac{U_2/\nu_2}{U_1/\nu_1} \overset{d}{=} F_{\nu_2,\nu_1}$

Inverse cdf of F-distribution is tabulated: (Notes 202=pp246-247)

$Z \overset{d}{=} F_{8,12} \Rightarrow \Pr(Z < 2.85) = 0.95$, & $\Pr(Z < 4.50) = 0.99$

$\frac{1}{Z} \overset{d}{=} F_{12,8} \Rightarrow \Pr(\frac{1}{Z} < 3.28) = 0.95 \Rightarrow \Pr(Z > 0.305) = 0.95$

Therefore, $\Pr(0.305 < Z < 2.85) = 0.90$

and similarly, $\Pr(0.238 < Z < 3.51) = 0.95$.

Lower quantiles can be obtained using $c_q(F_{\nu_2,\nu_1}) = 1/c_{1-q}(F_{\nu_1,\nu_2})$; so only the upper quantiles are tabulated.

In MINITAB and EXCEL F-quantiles are easily obtained.

**Application of the F-distribution: comparison of variances**

If $\frac{(n_1 - 1)S_1^2}{\sigma_1^2} \overset{d}{=} \chi^2_{n_1 - 1}$, $\frac{(n_2 - 1)S_2^2}{\sigma_2^2} \overset{d}{=} \chi^2_{n_2 - 1}$, and independent

then $\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \overset{d}{=} F_{n_1 - 1, n_2 - 1}$

This result can be used for inference on the variance ratio: $\frac{\sigma_1^2}{\sigma_2^2}$
example  \( n_1 = 9, n_2 = 13: \) \( \frac{S_1^2}{\sigma_1^2} \overset{d}{=} F_{8,12} \)

\[ \begin{align*}
\therefore \Pr & \left( 0.238 < \frac{S_1^2}{\sigma_1^2} \frac{S_2^2}{\sigma_2^2} < 3.51 \right) = 0.95 \\
\therefore \Pr & \left( 0.238 \frac{S_1^2}{S_2^2} < \frac{\sigma_1^2}{\sigma_2^2} < 3.51 \frac{S_2^2}{S_1^2} \right) = 0.95 \\
\therefore \Pr & \left( 0.284 \frac{S_1^2}{S_2^2} < \frac{\sigma_1^2}{\sigma_2^2} < 4.20 \frac{S_2^2}{S_1^2} \right) = 0.95
\end{align*} \]

which specifies a confidence interval for the ratio of the variances.

example Construct a test of size 0.05 for \( H_0: \sigma_1^2 = \sigma_2^2 \) against \( H_1: \sigma_1^2 \neq \sigma_2^2 \) using samples of \( n_1 = 25 \) and \( n_2 = 10 \).

decision rule: reject \( H_0 \) unless \( a < \frac{S_1^2}{S_2^2} < b \)
distribution: \( \frac{S_1^2}{S_2^2} \frac{\sigma_2^2}{\sigma_1^2} \overset{d}{=} F_{24,9} \)

size = 0.05: \( \Pr \left( a < \frac{S_1^2}{S_2^2} < b \mid \sigma_1^2 = \sigma_2^2 \right) = 0.95 \)

So, reject \( H_0 \) unless 0.371 < \( \frac{S_1^2}{S_2^2} < 3.61 \).

**Comparison of means**

1. **variances known**

\[ \bar{X}_1 - \bar{X}_2 \overset{d}{=} N \left( \mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} \right) \]

So, this is equivalent to \( Y \overset{d}{=} N(\mu, \frac{\sigma^2}{n_0}) \), where \( \sigma^2 \) is known, which is a simple problem that we know how to solve.

example  \( n_1 = 25 \quad \bar{x}_1 = 11.43 \quad (\sigma_1^2 = 4.0) \)

\( n_2 = 10 \quad \bar{x}_2 = 9.74 \quad (\sigma_2^2 = 2.5) \)

Then we have \( X_1 - X_2 \overset{d}{=} N(\mu_1 - \mu_2, 0.41) \);
and so a 95% CI for \( \mu_1 - \mu_2 \) is \( 1.69 \pm 1.96\sqrt{0.41} \),
i.e. \( 0.43 < \mu_1 - \mu_2 < 2.95 \)

**exercise** Check that a test of size 0.01 for these samples to test \( H_0: \mu_1 = \mu_2 \) vs \( H_1: \mu_1 \neq \mu_2 \) is given by:

“reject \( H_0 \) if \( |\bar{x}_1 - \bar{x}_2| > 1.65 \).”

It is not often that the variances are known, but this result is useful as a large sample approximation:

\[ \begin{align*}
X_1 & \overset{d}{=} \text{Bi}(100, \theta_1) \quad \Rightarrow \quad \frac{X_1}{100} \approx N \left( \theta_1, \frac{\theta_1(1 - \theta_1)}{100} \right) \\
X_2 & \overset{d}{=} \text{Bi}(60, \theta_2) \quad \Rightarrow \quad \frac{X_2}{60} \approx N \left( \theta_2, \frac{\theta_2(1 - \theta_2)}{60} \right)
\end{align*} \]

Thus an approximate 95% CI for \( \theta_1 - \theta_2 \) is given by

\[ \hat{\theta}_1 - \hat{\theta}_2 \pm 1.96 \sqrt{\frac{\hat{\theta}_1(1 - \hat{\theta}_1)}{100} + \frac{\hat{\theta}_2(1 - \hat{\theta}_2)}{60}}. \]

For example if \( x_1 = 54 \) and \( x_2 = 27 \), so that \( \hat{\theta}_1 = 0.54 \) and \( \hat{\theta}_2 = 0.45 \), then the 95% CI is \( -0.07 < \theta_1 - \theta_2 < 0.25 \).
2. variances unknown but equal
\[ \frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{\sigma \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \overset{d}{=} N(0, 1) \]

By analogy with the one sample case, we might hope that replacement of \( \sigma \) by \( S \) would result in a \( t \) distribution.

But what \( S \)? . . . and what \( t \)?
\[ \frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{S \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \overset{d}{=} t_{n_1+n_2-2} \]

where \( S^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1 + n_2 - 2} \)

example \( n_1 = 25 \) \( \bar{x}_1 = 11.43 \) \( s_1^2 = 3.79 \)
\( n_2 = 10 \) \( \bar{x}_2 = 9.74 \) \( s_2^2 = 2.21 \)
Test the hypothesis \( H_0: \mu_1 = \mu_2 \) vs \( H_1: \mu_1 \neq \mu_2 \)

Assumptions:
1. samples random (iidrvs)
2. samples independent
3. populations normally distributed
4. population variances equal

test statistic: \( T = \frac{\bar{X}_1 - \bar{X}_2}{S \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \)

decision rule: reject \( H_0 \) if \( |T| > c \)
distribution:
\[ \frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{S \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \overset{d}{=} t_{n_1+n_2-2} \]

thus \( H_0 \Rightarrow T \overset{d}{=} t_{n_1+n_2-2} \)

size = 0.05 \( \Rightarrow \) \( Pr(|T| > c \mid H_0) = 0.05 \)
and \( H_0 \Rightarrow T \overset{d}{=} t_{33} \), so \( c = c_{0.975}(t_{33}) = 2.034 \).

Thus the test is to reject \( H_0 \) if \( |T| > 2.034 \).

The observations give \( t = \frac{1.69}{1.833 \times 0.3742} = 2.464 \),
and so we reject \( H_0 \).

Alternatively, \( P = 2 \Pr(t_{33} > 2.464) \approx 0.02 \).

3. Variances unknown and unequal
\[ \frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \overset{d}{=} t_k, \]

where \( \min(n_1 - 1, n_2 - 1) \leq k \leq n_1 + n_2 - 2. \)

The formula for \( k \) is given in the notes: p.76.

(Note: It is simply obtained by finding \( k \) such that \( \chi^2_k \) has the same mean and variance as \( \frac{s_1^2}{n_1} + \frac{s_2^2}{n_2} \).)
MINITAB implementation:

MTB > TWOSAMPLE C1 C2;
SUBC> POOL.

If the POOL subcommand is used the pooled variance estimate is used — on the assumption that the population variances are equal.
If no subcommand is used the variances are assumed unequal and the above approximate test is used.

Comparison of means of \( k \) normal populations

<table>
<thead>
<tr>
<th>population</th>
<th>sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X_1 \overset{d}{=} N(\mu_1, \sigma^2) )</td>
<td>( X_{11}, X_{12}, \ldots, X_{1n} )</td>
</tr>
<tr>
<td>( X_2 \overset{d}{=} N(\mu_2, \sigma^2) )</td>
<td>( X_{21}, X_{22}, \ldots, X_{2n} )</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>( X_k \overset{d}{=} N(\mu_k, \sigma^2) )</td>
<td>( X_{k1}, X_{k2}, \ldots, X_{kn} )</td>
</tr>
</tbody>
</table>

Note 1: we assume equal variances
Note 2: for now, we assume equal sample sizes.

Test \( H_0: \mu_1 = \mu_2 = \cdots = \mu_k \) vs \( H_1: H_0 \).

| \( X_i \overset{d}{=} N(\mu_i, \sigma^2) \) | \( \bar{X}_i \overset{d}{=} N(\mu_i, \frac{\sigma^2}{n}) \) | \( \frac{(n-1)S_i^2}{\sigma^2} \overset{d}{=} \chi_{n-1}^2 \) |
| \hline |
| If \( H_0 \) true, then: |
| \( X \overset{d}{=} N(\mu, \sigma^2) \) | \( \bar{X} \overset{d}{=} N(\mu, \frac{\sigma^2}{N}) \) | \( \frac{(N-1)S_T^2}{\sigma^2} \overset{d}{=} \chi_{N-1}^2 \) |
| \( \bar{X} \overset{d}{=} N(\mu, \frac{\sigma^2}{n}) \) | \( \bar{X} \overset{d}{=} N(\mu, \frac{\sigma^2}{N}) \) | \( \frac{(N-1)S_T^2}{\sigma^2} \overset{d}{=} \chi_{N-1}^2 \) |

So, we have:

- total SS \( T = (N-1)S_T^2 \overset{d}{=} \sigma^2 \chi_{N-1}^2 \) \( H_0 \)
- within SS \( W = \sum (n-1)S_i^2 \overset{d}{=} \sigma^2 \chi_{N-k}^2 \) always
- between SS \( B = n(k-1)S_B^2 \overset{d}{=} \sigma^2 \chi_{k-1}^2 \) \( H_0 \)

Further, we find that: \( T = W + B \)

“Analysis of variance”: the variance is analysed into two components — one attributable to within group spread and the other to the between group spread (the spread of the means).

\( H_0 \) true:

\( \bullet \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \) [1] \( \bullet \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \) 

\( \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \) [2] \( \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \) 

\( \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \) [3] 

\( \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \) [4] 

\( \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \) [5] 

\( \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \) [6] 

\( \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \) [7] 

\( \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \) [8] 

\( \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \_ \) [9] 

We see that if \( H_0 \) is not true then both \( B \) & \( T \) tend to be larger.
We have \( T = W + B \)
and if \( H_0 \) true, then:
\[
\sigma^2 \chi^2_{N-1} = \sigma^2 \chi^2_{N-k} + \sigma^2 \chi^2_{k-1}
\]
Hence, if \( H_0 \) is true then
\[
F = \frac{B/(k-1)}{W/(N-k)} \Rightarrow F_{k-1,N-k}
\]
and if \( H_0 \) is not true then \( F \) tends to be large.

Thus a test of \( H_0 \) is given by the decision rule: reject \( H_0 \) if \( F > c \)
If this test is to have size = 0.05, then \( c = c_{0.95}(F_{k-1,N-k}) \)

Despite the ad hoc way in which the test has been derived here, it is in fact the best test of this null hypothesis: it is the Likelihood Ratio Test for this situation (proved later).

**analysis of variance**

<table>
<thead>
<tr>
<th></th>
<th>df</th>
<th>SS</th>
<th>MS</th>
<th>( F )</th>
</tr>
</thead>
<tbody>
<tr>
<td>between</td>
<td>( k-1 )</td>
<td>( B )</td>
<td>( B/(k-1) )</td>
<td>( B/(k-1) )</td>
</tr>
<tr>
<td>within</td>
<td>( N-k )</td>
<td>( W )</td>
<td>( W/(N-k) )</td>
<td>( W/(N-k) )</td>
</tr>
<tr>
<td>total</td>
<td>( N-1 )</td>
<td>( T )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

decision rule: reject \( H_0 \) if \( F_{obs} > c_{0.95}(F_{k-1,N-k}) \).

**P-value:** \( P = Pr(F_{k-1,N-k} > F_{obs}) \).

**example**

<table>
<thead>
<tr>
<th></th>
<th>df</th>
<th>SS</th>
<th>MS</th>
<th>( F )</th>
</tr>
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<tbody>
<tr>
<td>between</td>
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<td>3.00</td>
<td>1.00</td>
<td>6.75</td>
</tr>
<tr>
<td>within</td>
<td>11</td>
<td>1.62</td>
<td>0.15</td>
<td></td>
</tr>
<tr>
<td>total</td>
<td>14</td>
<td>4.62</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Tables:** \( c_{0.95}(F_{3,11}) = 3.59 \), \( c_{0.99}(F_{3,11}) = 6.22 \).

So we would reject \( H_0 \).

<table>
<thead>
<tr>
<th></th>
<th>( n_i )</th>
<th>( x_{i\bullet} )</th>
<th>( s_i^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.47, 2.59, 3.01, 2.17</td>
<td>4</td>
<td>10.24</td>
<td>0.1212</td>
</tr>
<tr>
<td>2.93, 3.56, 3.52</td>
<td>3</td>
<td>10.01</td>
<td>0.1245</td>
</tr>
<tr>
<td>2.47, 1.93, 2.20</td>
<td>3</td>
<td>6.60</td>
<td>0.0729</td>
</tr>
<tr>
<td>3.14, 3.29, 3.97, 2.94, 2.76</td>
<td>5</td>
<td>16.10</td>
<td>0.2159</td>
</tr>
<tr>
<td>( N = 15 )</td>
<td>( \bar{x}_{\bullet} = 42.95 )</td>
<td>( s_T^2 = 0.3299 )</td>
<td></td>
</tr>
</tbody>
</table>

\[
\sum \sum x_{ij}^2 = 2.47^2 + 2.59^2 + \ldots + 2.76^2 = 127.5985
\]

\[
\sum \frac{s_i^2}{n_i} = \frac{10.24^2}{4} + \ldots + \frac{16.10^2}{5} = 125.9764
\]

\[
\frac{T^2}{N} = \frac{42.95^2}{15} = 122.9802
\]

**Computation:**

\[
T = \sum \sum (x_{ij} - \bar{x})^2 = \sum \sum x_{ij}^2 - \frac{\sum \sum x_{ij}^2}{N} = (N-1)s_T^2
\]

\[
B = \sum n_i (\bar{x}_i - \bar{x})^2 = \sum \frac{s_i^2}{n_i} - \frac{\sum s_i^2}{N} = T - W
\]

\[
W = \sum \sum (x_{ij} - \bar{x}_{i\bullet})^2 = T - B = \sum (n_i - 1)s_i^2
\]
...or better, use a computer!

MTB > print c1-c4

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<th>C1</th>
<th>C2</th>
<th>C3</th>
<th>C4</th>
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<td>2</td>
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<td>5</td>
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MTB > aovo c1-c4

ANALYSIS OF VARIANCE

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<th>SOURCE</th>
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INDIVIDUAL 95 PCT CI'S FOR MEAN
BASED ON POOLED STDEV

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<tr>
<th>N</th>
<th>MEAN</th>
<th>STDEV</th>
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<td>C1</td>
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<td>2.5600</td>
</tr>
<tr>
<td>C2</td>
<td>3</td>
<td>3.3367</td>
</tr>
<tr>
<td>C3</td>
<td>3</td>
<td>2.2000</td>
</tr>
<tr>
<td>C4</td>
<td>5</td>
<td>3.2200</td>
</tr>
</tbody>
</table>

POOLED STDEV = 0.3840

\[ \bar{x}_1 = 2.56, \bar{x}_2 = 3.34, \bar{x}_3 = 2.20, \bar{x}_4 = 3.22; \quad s^2 = 0.1475 \]

95% CI for \( \sigma^2 \) based on \( \frac{11S^2}{\sigma^2} \) \( \d \chi^2_{11} \)

\[ \Pr(3.816 < \frac{11S^2}{\sigma^2} < 21.92) = 0.95 \quad \Rightarrow \quad 0.074 < \sigma^2 < 0.425 \]

(0.272 < \sigma < 0.652)

95% CI for \( \mu_1 \) based on \( \frac{\bar{X}_1 - \mu_1}{S/\sqrt{4}} \) \( \d t_{11} \)

\[ \Pr(-2.201 < \frac{\bar{X}_1 - \mu_1}{\sqrt{\frac{1}{4}S^2}} < 2.201) = 0.95 \quad \Rightarrow \quad 2.14 < \mu_1 < 2.98 \]

95% CI for \( \mu_1 - \mu_2 \) based on \( \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{S/\sqrt{\frac{1}{4} + \frac{1}{3}}} \) \( \d t_{11} \)

\[ \Pr(-2.201 < \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{7}{12}S^2}} < 2.201) = 0.95 \]

\[ \Rightarrow \quad -1.43 < \mu_1 - \mu_2 < -0.13 \]

95% PI for \( X_1 \) based on \( \frac{X_1^* - \bar{X}_1}{S/\sqrt{1 + \frac{1}{4}}} \) \( \d t_{11} \)

\[ \Pr(-2.201 < \frac{X_1^* - \bar{X}_1}{\sqrt{\frac{7}{12}S^2}} < 2.201) = 0.95 \quad \Rightarrow \quad 1.61 < X_1^* < 3.51 \]
Multiple comparisons

Suppose we observe the following sample which is supposed to be from a standard normal population:

\[ 0.73 \quad -1.48 \quad 0.07 \quad 2.17 \quad -0.67 \]

We look at the fourth observation and note that:

\[ \Pr(X \geq 2.17) = 0.015 \]

Does this mean that we should reject the model?

\[ \Pr(X_{(5)} \geq 2.17) = 1 - 0.985^5 = 0.073 \]

So if we picked out 2.17 because it was the biggest, this observation is not significant. But ...

Similarly \( x_4 - x_2 = 3.65 \) which is large when compared to \( N(0, 2) \), but is it large when compared against the distribution of \( R = X_{(5)} - X_{(1)} \)?

**Answer:** No. It would need to be greater than 3.86 to be significant at the 0.05-level. *(from Tables p245; \( k = 5, \nu = \infty \))

**Post-hoc procedures (Tukey)**

standardised range distribution: \( Q_{k, \nu} = \frac{R_k}{S_\nu} \)

where \( R_k \) is the range of a sample of \( k \) \( N(\mu, \sigma^2) \) random variables;

and \( S_\nu \) is an estimator of \( \sigma \) having \( \nu \) degrees of freedom which is independent of \( R_k \).

Tables of standardised range distribution (page 233)

\[ \Pr(Q_{k, \nu} < c) = 0.95 \]
\[ \Pr(R_k < cS_\nu) = 0.95 \]

So if we observe \( |Z_i - Z_j| > cS_\nu \), then this can be taken as evidence that \( E(Z_i) \neq E(Z_j) \).

\[ \Pr(R_k < cS) = 0.95 \]
\[ \Pr(-cS < Z_i - Z_j < cS) = 0.95 \]
for all \( i, j \)

Now, \( \bar{X}_i - \mu_i \overset{d}{=} N(0, \frac{\sigma^2}{n}) \), so:

\[ \Pr \left( -c\frac{S}{\sqrt{n}} < (\bar{X}_i - \mu_i) - (\bar{X}_j - \mu_j) < c\frac{S}{\sqrt{n}} \right) = 0.95 \]
for all \( i, j \)

\[ \Pr \left( \bar{X}_i - \bar{X}_j - c\frac{S}{\sqrt{n}} < \mu_i - \mu_j < \bar{X}_i - \bar{X}_j + c\frac{S}{\sqrt{n}} \right) = 0.95 \]
for all \( i, j \)

where \( c = c_{0.95}(Q_{k, \nu}) \).

This gives a 95% simultaneous confidence interval for all the differences between means. These are the Tukey intervals provided by MINITAB.

If we observe \( |\bar{X}_i - \bar{X}_j| \) greater than \( cS/\sqrt{n} \), then this is evidence that \( \mu_i \) and \( \mu_j \) are significantly different.

The quantity \( cS/\sqrt{n} \) is often called the Least Significant Difference (LSD).
One minor problem: $c \frac{s}{\sqrt{n}}$ is not in the form $c \text{se}(\bar{X}_i - \bar{X}_j)$, since $\text{var}(\bar{X}_i - \bar{X}_j) = \frac{2\sigma^2}{n}$. Thus $c \frac{s}{\sqrt{n}} = \frac{c}{\sqrt{2}} \frac{s}{\sqrt{n}} = \frac{c}{\sqrt{2}} \text{se}(\bar{X}_i - \bar{X}_j)$.

It is more of a problem when there are groups of unequal sizes, since then this result applies only as an approximation: but we still use the multiplier $c/\sqrt{2}$, where $c = c_{0.95}(Q_k, \nu)$.

$$\text{LSD} \approx \frac{c}{\sqrt{2}} \sqrt{\frac{1}{n_i} + \frac{1}{n_j}}$$

```
MTB > stack c1-c4 c10;
SUBC> subs c11.
MTB > oneway c10 c11;
SUBC> fisher;
SUBC> tukey.
```

**Analysis of Variance on C10**

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>p</th>
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<tr>
<td>C11</td>
<td>3</td>
<td>2.996</td>
<td>0.999</td>
<td>6.77</td>
<td>0.007</td>
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<tr>
<td>Error</td>
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<td>1.622</td>
<td>0.147</td>
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<tr>
<td>Total</td>
<td>14</td>
<td>4.618</td>
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**Individual 95% CIs For Mean Based on Pooled StDev**

<table>
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<tr>
<th>N</th>
<th>Mean</th>
<th>StDev</th>
<th>CIs</th>
</tr>
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<tbody>
<tr>
<td>1</td>
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<td>2.5600</td>
<td>0.3481</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>3.3367</td>
<td>0.3528</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>2.2000</td>
<td>0.2700</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>3.2200</td>
<td>0.4647</td>
</tr>
</tbody>
</table>

Pooled StDev = 0.3840 1.80 2.40 3.00 3.60

**Fisher's pairwise comparisons**

- Family error rate = 0.183
- Individual error rate = 0.050
- Critical value = 2.201

<table>
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<th>2</th>
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<tbody>
<tr>
<td>2</td>
<td>-1.4222 -0.1311</td>
</tr>
<tr>
<td>3</td>
<td>-0.2855 0.4466</td>
</tr>
<tr>
<td>4</td>
<td>-1.2270 -0.5006 -1.6372</td>
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<tr>
<td></td>
<td>-0.0930 0.7339 -0.4028</td>
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</tbody>
</table>

**Tukey's pairwise comparisons**

- Family error rate = 0.050
- Individual error rate = 0.012
- Critical value = 4.26

<table>
<thead>
<tr>
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<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>-1.6601 0.1068</td>
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<tr>
<td>3</td>
<td>-0.5235 0.1922</td>
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<tr>
<td>4</td>
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<td>0.1160 0.9614 -0.1752</td>
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