Topic 4: Linear Programming

The Simplex method

Let us revisit the 'Shirts and Jackets' problem. The mathematical problem was

Maximise

\[ P = 5x_1 + 6x_2 \]

\[ \text{subject to the constraints} \]

\[ 4x_1 + 2x_2 \leq 2000 \quad (\text{problem constraint}) \]
\[ x_1 + 3x_2 \leq 1500 \quad (\text{problem constraint}) \]
\[ x_1 \geq 0 \]
\[ x_2 \geq 0 \quad (\text{non-negativity constraints}) \]

Method 1: Graphical solution (already covered)

We obtained the answer

\[ \max P = 3900 \text{ when } (x_1, x_2) = (300, 400) \]

The answer occurred on the boundary
of the feasible region. Moreover it occurred at a corner point (feasible intersection points).

We would like to solve this problem without drawing a graph. We need a method to treat problems in which there are more than two decision variables. We do not have a general method for solving a set of inequations but we do have a method for solving linear equations (i.e., row reduction).

Method 2: Solve linear equations to give all intersection points.

Painful but necessary to understand simplex.

Do without a graph.
Let us find all intersections of \( \mathbb{Q} \).

The lines \( x_1 = 0, \ x_2 = 0, \ 4x_1 + 2x_2 = 2000 \) and \( x_1 + 3x_2 = 1500 \)

We take 2 lines at a time (2 variables \( \Rightarrow \) 2 equations).

Then will be \( \binom{4}{3} = \frac{4 \times 3}{2 \times 1} = 6 \) intersection points.

Before we do this let us rewrite the problem constraints (inequalities) as an underdetermined system of equations by introducing a new variable for each inequality:

Thus

\[
4x_1 + 2x_2 \leq 2000
\]
\[
x_1 + 3x_2 \leq 1500
\]

become

\[
4x_1 + 2x_2 + s_1 = 2000
\]
\[
x_1 + 3x_2 + s_2 = 1500
\]
We have introduced two new variables $s_1$ and $s_2$. To take up the 'slack' between the value of the RHS and LHS of the inequalities. The new variables are called slack variables.

Our new system of constraints become

$$4x_1 + 2x_2 + s_1 = 2000$$
$$x_1 + 3x_2 + s_2 = 1500$$

with $x_1 \geq 0$, $x_2 \geq 0$, $s_1 \geq 0$, $s_2 \geq 0$.

Why do this? We can find the 6 intersection points by setting two of the four variables in our new problem constraints equations to zero. That is, in

$$4x_1 + 2x_2 + s_1 = 2000 \quad (\ast)$$
$$x_1 + 3x_2 + s_2 = 1500$$

and solve. OK, but why this way...
(i) Put $S_1 = 0$ and $S_2 = 0$ in (*)

(ie: zero slack in both equations)

Then we have

\[ 4x_1 + 2x_2 = 2000 \]
\[ x_1 + 3x_2 = 1500 \]

The solution is \( x_1 = 300, \ x_2 = 400 \)

with \( S_1 = 0, \ S_2 = 0 \)

Question: Are all the variable values non-negative?

Answer: Yes so ... big tick \( \checkmark \)

Why ask this? \( \rightarrow \) Tell us whether intersection point is feasible without recourse to a graph!!!
(ii) Put $s_1 = 0$ and $x_1 = 0$ in $(\ast)$

Then
\[
\begin{align*}
2x_2 &= 2000 \\
3x_2 + s_2 &= 1500
\end{align*}
\]

\[
\begin{align*}
x_1 &= 0 \\
x_2 &= 1000 \\
s_1 &= 0 \\
s_2 &= -1500
\end{align*}
\]

\[\text{\(\times\)}\]

(iii) Put $s_1 = 0$ and $x_2 = 0$ in $(\ast)$

Then
\[
\begin{align*}
4x_1 + x_2 &= 2000 \\
x_1 + s_2 &= 1500
\end{align*}
\]

\[
\begin{align*}
x_1 &= 500 \\
x_2 &= 0 \\
s_1 &= 0 \\
s_2 &= 1000
\end{align*}
\]

\[\text{\(\checkmark\)}\]

(iv) Put $s_2 = 0$ and $x_1 = 0$ in $(\ast)$

Then
\[
\begin{align*}
2x_2 + s_1 &= 2000 \\
3x_2 &= 1500
\end{align*}
\]

\[
\begin{align*}
x_1 &= 0 \\
x_2 &= 500 \\
s_1 &= 1000 \\
s_2 &= 0
\end{align*}
\]

\[\text{\(\checkmark\)}\]

(v) Put $s_2 = 0$ and $x_2 = 0$ in $(\ast)$

Then
\[
\begin{align*}
4x_1 + s_1 &= 2000 \\
x_1 &= 1500
\end{align*}
\]

\[
\begin{align*}
x_1 &= 1500 \\
x_2 &= 0 \\
s_1 &= -4000 \\
s_2 &= 0
\end{align*}
\]

\[\text{\(\times\)}\]
Put $x_1=0$ and $x_2=0$ in $\odot$

Then $S_1 = 2000 \begin{cases} x_1=0 \\ x_2=0 \\ S_1=2000 \\ S_2=1500 \end{cases}$ $\blacksquare$

All these 'solutions' (i) to (vi), are called basic solutions of $\odot$ (regardless of whether they are labelled $\blacksquare$ or $\times$).

Those labelled $\blacksquare$ are basic feasible solutions.

Those labelled $\times$ are basic infeasible solutions.

To solve the problem we now evaluate $P = 5x_1 + 6x_2$ for each of the basic feasible solutions $\blacksquare$ and choose the one with the largest value of $P$. 
\[ x_1 = 0 \]
\[ x_2 = 500 \]
\[ S_1 = 1000 \]
\[ S_2 = 0 \]

\[ x_1 = 0 \]
\[ x_2 = 1000 \]
\[ S_1 = 0 \]
\[ S_2 = -1500 \]

\[ x_1 = 300 \]
\[ x_2 = 400 \]
\[ S_1 = 0 \]
\[ S_2 = 0 \]

\[ x_1 = 0 \]
\[ x_2 = 0 \]
\[ S_1 = 2000 \]
\[ S_2 = 1000 \]

\[ x_1 = 500 \]
\[ x_2 = 0 \]
\[ S_1 = 0 \]
\[ S_2 = 1000 \]

\[ x_1 = 1500 \]
\[ x_2 = 0 \]
\[ S_1 = -4000 \]
\[ S_2 = 0 \]

\[ p = 3000 \]
\[ x_1 + 3x_2 = 15 \]
\[ x_2 = 2000 \]
\[ x_1 = 500 \]

\[ p = 3900 \]
\[ p = 2500 \]

\[ p \rightarrow \text{all variables not negative} \]

\[ x \rightarrow \text{some variables have negative values} \]
Thus we can solve this problem without drawing a graph.

OK but while this method works it is impractical for large problems.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Graph</th>
<th>Task</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 decision variables</td>
<td><img src="image1" alt="Graph" /></td>
<td>(4C_2 = 6) basic solns solve 6 pairs of eqns.</td>
</tr>
<tr>
<td>2 problem constraints</td>
<td><img src="image2" alt="Graph" /></td>
<td>(5C_3 = 10) basic solns</td>
</tr>
<tr>
<td>3 d.v</td>
<td><img src="image3" alt="Graph" /></td>
<td>(4C_4 = \frac{9 	imes 8 	imes 7 	imes 6 	imes 5}{4 	imes 3 	imes 2 	imes 1} = 126) basic solns</td>
</tr>
<tr>
<td>5 p.c</td>
<td><img src="image4" alt="Graph" /></td>
<td>?</td>
</tr>
</tbody>
</table>

There is a better way!

The (wonderful) **SIMPLEX METHOD** (hooray!)

George Danzig (1947)
Method 3 (SIMPLEX)

We rewrite the problem as

Maximise $P$

where

$$\begin{align*}
4x_1 + 2x_2 + s_1 &= 2000 \\
x_1 + 3x_2 + s_2 &= 1500 \\
-5x_1 - 6x_2 &+ P = 0
\end{align*}$$

subject to the constraints $x_1, x_2, s_1, s_2 \geq 0$.

1. Simplex idea (Standard maximisation problem)

It is clear above that $x_1 = 0, x_2 = 0$ gives us one feasible solution, (no need for a graph)

$$\begin{align*}
x_1 &= c \\
x_2 &= c \\
s_1 &= 200 \\
s_2 &= 150 \\
P &= c
\end{align*}$$

Of course $P = 0$ is not the maximum value of $P$, just a place to start. A basic feasible solution
$S_1$ and $S_2$: basic variables
$x_1$ and $x_2$: non-basic variables ($=0$)

for this basic feasible solution.

Flow chart:

1. A basic feasible solution
2. Method to find another basic feasible solution
3. A better basic feasible solution
4. Test to see if it is the best possible
5. The best basic feasible solution

The method idea: Interchange basic and non-basic variables so that $P$ increases.
2. Entering variables

For every unit increase in \( x_1 \), \( P \) increases 5 while \( x_2 \) \( P \) increases 6. So let us choose \( x_2 \) to change from a non-basic variable to a basic variable \( (x_2^T) \).

Let it 'enter'. We call \( x_2 \) the entering variable. (Can choose either since both increase \( P \) but choose \( x_2 \) as the 'best' choice).

Now which variable, \( S_1 \) or \( S_2 \), are we going to change from a basic to a non-basic variable (the exiting variable)?

(a) The \( S_2 \) equation tells us that if \( x_1 = 0 \) \( x_2 \) can be increased to 1000 before \( S_1 \) becomes zero. But if we put \( x_2 = 1000 \) into the \( S_2 \) equation...
(with $x_1 = 0$) Then we have $S_2 < 0$ which is not allowed.

(b) The $S_2$ equation tells us that if we keep $x_1 = 0$, we can increase $x_2$ to 500 before $S_2$ becomes zero. And if we put $x_1 = 0$, $x_2 = 500$ into the $S_1$ equation then $S_1$ is still positive.

So we must choose the basic variable $s_2$ to become non-basic, since that gives the smallest increase in $x_2$ and so ensures all the other basic variables stay positive.

In a nutshell we choose $s_2$ since

in (a) $\frac{2000}{2} = 1000$ while in (b) $\frac{1500}{3} = 500$

\[ \uparrow \quad \text{(larger)} \]

RHS

\[ \text{coefficient of entering variable} \]

So $S_2$ is the exiting variable.
Summary so far:

We started at one basic feasible solution with $x_1, x_2$ as non-basic variables ($= 0$)
$s_1, s_2$ as basic variables ($\geq 0$)

We chose $x_2$ to be the entering variable: That is, go from non-basic to basic since the coefficient of $x_2$ in the 'P' equation was the most negative.

We chose $s_2$ to be the exiting variable: That is, go from basic to non-basic since the quotient of the RHS and the coefficient of the entering variable ($x_2$) in the equation containing $s_2$ was smallest, so ensuring all our variables stay non-negative (our new solution is feasible)
(We only consider coefficients of entering variables that are positive!)
We will now transform the equations to give \( x_1, s_2 \) as non-basic \((= 0)\)
and \( s_1, x_2 \) as basic \((\geq 0)\)
variables so as to give another basic feasible solution (a better one?)

We use a combination of (B) and (C) row operations (called a pivot) to enable us to see the new feasible solution easily.

The new equations are row equivalent so give the same solution set but we choose to specify different variables to be zero so 'pick' out a different solution from the underdetermined set.
Basic feasible solution

New solution
To enact our 'pivot' we write our initial equations as a 'Simplex Tableaux'.

3. Simplex Tableaux

Let us write our equations in matrix form

\[
\begin{bmatrix}
  x_1 & x_2 & s_1 & s_2 & P & \text{RHS} \\
  4 & 2 & 1 & 0 & 0 & 2000 \\
  1 & 3 & 0 & 1 & 0 & 1500 \\
  -5 & -6 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

We begin with \( s_1 \) and \( s_2 \) basic (you can read off their values if \( x_1 = x_2 = 0 \)).

\[ \rightarrow \text{Find a negative number in the last row (use most negative): This gives us our entering variable. We call this column of the Tableaux our 'Pivot column'.} \]

\[ \rightarrow \text{Calculate quotients (only positive coeff) and find smallest: This gives our exiting variable (pivot row).} \]
Pivot element

4. Obtaining a 'better' basic feasible soln.

We now use row operations, first a (B) operation, to get a "1" where the pivot element resides (i.e., pivot column); then secondly we use 2 (C) operations to obtain "0"s in the rest of that column.

Doing these row operations is known as doing a 'Pivot'.

What happens to the $S_2$ column?

Clearly $S_1$ and $P$ columns are unchanged.
\[
\begin{bmatrix}
4 & 2 & 1 & 0 & 0 & \text{RHS} \\
1 & 3 & 0 & 1 & 0 & \text{2000} \\
-5 & -6 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\[
\frac{1}{3}R_2 \rightarrow R_2 \quad \text{(B)}
\]

\[
\begin{bmatrix}
4 & 2 & 1 & 0 & 0 & \text{RHS} \\
\frac{1}{3} & 1 & 0 & \frac{1}{3} & 0 & \text{2000} \\
-5 & -6 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\[-2R_2 + R_1 \rightarrow R_1 \quad \text{500} \]

\[
\begin{bmatrix}
\frac{10}{3} & 0 & 1 & -\frac{2}{3} & 0 & \text{RHS} \\
\frac{1}{3} & 1 & 0 & \frac{1}{3} & 0 & \text{1000} \\
-3 & 0 & 0 & 2 & 1 & \text{500} \\
\end{bmatrix}
\]

\[6R_2 + R_3 \rightarrow R_3 \]

\[(C) \text{Twice} \]

We have \( \text{bivoted} \).

\[ \rightarrow \text{Same information as before} \]

eg: bottom eqn: \( P = 3000 - 2S_2 + 3x_1 \)

If \( x_1 = 0, S_2 = 1500 \) then \( P = 0 \) but we consider \( x_1 = S_2 = 0 \): This eqn is easier to see than \( P = 3000 \).
\[
\begin{bmatrix}
10/3 & 0 & 1 & -\frac{23}{3} & 0 & \text{RHS}
\end{bmatrix}
\]

\[
\begin{bmatrix}
1/3 & 0 & 0 & \frac{1}{3} & 0 & \text{RHS}
\end{bmatrix}
\]

\[
\begin{bmatrix}
-3 & 0 & 0 & 2 & 1 & \text{RHS}
\end{bmatrix}
\]

\[
\begin{array}{c}
\text{basic} \\
\text{variables}
\end{array}
\quad
\begin{array}{c}
\text{non-basic} \\
\text{variables}
\end{array}
\]

Setting \( x_1 = 0 \) and \( s_2 = 0 \), we have

\[ s_1 = 1000, \]

\[ x_2 = 500, \]

and \( p = 3000 \)

This is our new basic feasible solution. It is better than our first \((3000 > 0)\).

The above tableaux represents a system of equations equivalent to the first just rewritten to make our new basic feasible solution obvious.
5. Repeat the process if possible to find an even better solution.

Since \( P = 3000 - 2S_2 + 3x \), we would like to keep \( S_2 = 0 \) (as non-basic) and increase \( x \), (make basic) so as to increase \( P \). We 'see' this in the Tableaux (*) by noticing the negative number in the bottom row.

So we choose \( x_1 \), to become basic, that is, \( x_1 \), is the entering variable. We then calculate quotients to find the exiting variable.
This tells us that 300 is the most we can increase $x_1$ before one of the other (currently basic) variables goes negative. So we pivot on the element $\frac{10}{3}$ which makes $s_1$ our exiting variable.

\[
\begin{bmatrix}
\frac{10}{3} & 0 & 1 & -2\frac{1}{3} & 0 & 1000 \\
\frac{1}{3} & 1 & 0 & \frac{1}{3} & 0 & 500 \\
-3 & 0 & 0 & 2 & 1 & 3000
\end{bmatrix}
\xrightarrow{\frac{3}{10}R_1 \rightarrow R_1}
\begin{bmatrix}
1 & 0 & 3\frac{1}{10} & -\frac{1}{5} & 0 & 300 \\
\frac{1}{3} & 1 & 0 & \frac{1}{3} & 0 & 500 \\
-3 & 0 & 0 & 2 & 1 & 3000
\end{bmatrix}
\xrightarrow{-\frac{1}{3}R_1+R_2 \rightarrow R_2}
\begin{bmatrix}
1 & 0 & 3\frac{1}{10} & -\frac{1}{5} & 0 & 300 \\
0 & 1 & -\frac{1}{10} & 2\frac{1}{5} & 0 & 400 \\
0 & 0 & \frac{9}{10} & \frac{7}{5} & 1 & 3900
\end{bmatrix}
\xrightarrow{3R_1+R_3 \rightarrow R_3}
\begin{bmatrix}
1 & 0 & 3\frac{1}{10} & -\frac{1}{5} & 0 & 300 \\
0 & 1 & -\frac{1}{10} & 2\frac{1}{5} & 0 & 400 \\
0 & 0 & \frac{9}{10} & \frac{7}{5} & 1 & 3900
\end{bmatrix}
\]
$S_1$ and $S_2$ are now non-basic variables so we put $S_1 = 0$, $S_2 = 0$ and read off the values of the basic variables $x_1$ and $x_2$. Hence our new basic feasible solution is $S_1 = 0$, $S_2 = 0$, $x_1 = 300$, $x_2 = 400$ with $P = 3900$

6. The optimal solution.

We now notice that the bottom equation of (X'') says

$$P = 3900 - \frac{9}{10} S_1 - \frac{7}{5} S_2$$

which can only be decreased by increased either $S_1$ or $S_2$ since $S_1 > 0$, $S_2 > 0$.
This tells us that we have found the optimal solution (the best). That is, the answer to our problem.
(We see this in the Tableaux in the bottom row: it has only positive coefficients for non-basic variables)

Thus the maximum of P is 3900 when \((x_1, x_2) = (300, 400)\)

I. Two Important Observations
(a) In each step it is easy to read off the values of \(x_1, x_2, s_1, s_2, P\). The basic variables are (generally) tracked by coefficients of 1 and the non-basic variables are set to zero.
(b) We stop the procedure when there are no negative entries in the last row.
Introduction to Biomedical Mathematics

Simplex Procedure Summary

It is difficult to describe the simplex procedure precisely in a few words, and you may find the following instructions to be incomprehensible without an accompanying example. On page 312 of B & Z7 (page 308 B & Z6) there is a flow chart, which you might find to be a more understandable summary.

The Standard Maximum Problem - Summary of Procedure
(Selectiv Row Operations)

1. Write down the initial tableau.

2. Inspect the bottom row. If there is a negative entry, take the column for that entry as the pivot column. (There may be more than one negative entry, so a choice might have to be made. Choosing the negative entry of largest magnitude in the bottom row might lead to fewer steps in the procedure.)

3. In the chosen pivot column, check that there is at least one positive entry. Otherwise STOP - there is no optimal solution; the objective function is unbounded.

4. For each positive entry in the pivot column, calculate "the ratio" (divide the right hand element in the row by the positive entry), and take the pivot element to be the entry which yields the smallest ratio.

5. Make the pivot 1 by multiplication of the row by the appropriate constant.

6. By adding appropriate multiples of the pivot row to the other rows, create zeros in the pivot column, except for the pivot element itself.

7. Repeat steps 2. to 6. until there are no negative entries in the bottom row. Interpret the final tableau. Read off the maximum value of the objective function and the corresponding values of the decision variables.
Step 1:
Write the standard maximization problem in standard form, introduce slack variables to form the initial system, and write the initial tableau.

Step 2:
Are there any negative indicators in the bottom row?

Yes

Step 3:
Select the pivot column.

No

STOP
The optimal solution has been found.

Step 4:
Are there any positive elements in the pivot column above the dashed line?

Yes

STOP
The linear programming problem has no optimal solution.

No

Step 5:
Select the pivot element and perform the pivot operation.
The Simplex method provides a fast, usually efficient method of solving standard maximum LP problems.

The standard maximum LP problem is:

Maximise the objective function

\[ P = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \]

subject to the problem constraints of the form

\[ a_1 x_1 + a_2 x_2 + \ldots + a_n x_n \leq b \]

with \( b \geq 0 \)

and non-negativity constraints

\[ x_1 \geq 0, \ x_2 \geq 0, \ldots, \ x_n \geq 0 \]
Let us take a problem with \( n \) problem constraints. These can be written as

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq b_2 \\
    \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\leq b_m
\end{align*}
\]

with \( b_1 \geq 0, b_2 \geq 0, \ldots, b_m \geq 0 \)

So we have \( n \) decision variables with \( m \) problem constraints.

Note that these inequations can be written in matrix form

\[
Ax \leq b
\]

where \( A = [a_{ij}] \) is an \( m \times n \) matrix.
Let us define

\[ \mathbf{c}^T = \begin{bmatrix} c_1 & c_2 & \ldots & c_n \end{bmatrix} \]

\( \mathbf{c} \) is an \( 1 \times n \) row matrix

\[ \mathbf{U}_n = \begin{bmatrix} 0 & \ldots & 0 \end{bmatrix} \]

\( \mathbf{U}_n \) is an \( n \times 1 \) column matrix
Then we can rewrite our standard maximum LP problem as

Maximise \[ P = \xi^T \mathbf{x} \]

subject to \[ A \mathbf{x} \leq \mathbf{b} \quad \text{and} \quad \mathbf{x} \geq \mathbf{0} \]

with \[ \mathbf{b} \geq \mathbf{0} \mathbf{m} \].

Note that \( A \) is a matrix but \( P \) is simply a number (1x1 matrix).
To solve this problem we introduce $m$ slack variables $s_1, s_2, \ldots, s_m$ and rewrite our problem as a Simpler Tableaux:

\[
\begin{bmatrix}
A & I_m & \xi_m & b \\
-c^T & \xi^T_m & 1 & 0 \\
\end{bmatrix}
\]

Associated with initial non-basic variables being decision variables $x_1, \ldots, x_n$

Associated with initial basic variables (The slack variables $s_1, \ldots, s_m$)
Example

Maximise
\[ P = 10x_1 + 30x_2 \]

subject to
\[ 2x_1 + x_2 \leq 16 \]
\[ x_1 + x_2 \leq 12 \]
\[ x_1 + 2x_2 \leq 14 \]

with \( x_1 \geq 0 \) and \( x_2 \geq 0 \).

Rewritten in matrix notation we have

Maximise
\[ P = \begin{bmatrix} 10 & 30 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

subject to
\[ \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 16 \\ 12 \\ 14 \end{bmatrix} \]

with \[ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} c \\ e \end{bmatrix} \]
So

\[ c_j^T = \begin{bmatrix} 10 & 30 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \]

\[ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad b = \begin{bmatrix} 16 \\ 12 \\ 14 \end{bmatrix} \]

\[ z_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad z_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

Note that \( b \geq z_3 \) and \( m = 3, n = 2 \).

The initial Tableaux is

\[
\begin{bmatrix}
A & I_3 & c_3 & 1 \\
-2c^T & 0_3 & 1 & 0
\end{bmatrix}
\]

L:

\[
\begin{bmatrix}
2 & 1 & 1 & 0 & 0 & 1 & 10 \\
1 & 1 & 1 & 0 & 0 & 1 & 12 \\
1 & 2 & 0 & 0 & 1 & 0 & 14 \\
-10 & -30 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]
An allowed variation to the Simplex method:

When choosing the pivot column you may choose any column with a negative element in the bottom row. (No need to take most negative)

Return to previous example ('shirts + jackets')

Our initial Tableaux was

\[
\begin{bmatrix}
-5 & 2 & 1 & 0 & 0 & \text{RHS} \\
1 & 3 & 0 & 1 & 0 & 2000 \\
-3 & 0 & 0 & 1 & 0 & 1500
\end{bmatrix}
\]

This time we choose this column

We choose row 1 as the pivotrow since 500 < 1500

\$s_1, s_2 \text{ basic}\$
\[
\begin{array}{cccccc}
1 & \frac{3}{2} & \frac{1}{4} & 0 & 0 & 500 \\
1 & 3 & 0 & 1 & 0 & 1500 \\
-5 & -6 & 0 & 0 & 1 & 0
\end{array}
\]

\(-R_1 + R_2 \rightarrow R_2\)

\(5R_1 + R_3 \rightarrow R_3\)

\[
\begin{array}{cccccc}
1 & \frac{3}{2} & \frac{1}{4} & 0 & 0 & 500 \\
0 & \frac{5}{2} & -\frac{1}{4} & 1 & 0 & 1000 \\
0 & -\frac{7}{2} & \frac{5}{4} & 0 & 1 & 2500
\end{array}
\]

\(\frac{500}{\frac{5}{2}} = 1000\)

\(\frac{1000}{\frac{7}{2}} = 400\)

\(\frac{2}{5} R_2 \rightarrow R_2\)

\(\text{new pivot column}\)

\[
\begin{array}{cccccc}
1 & \frac{3}{2} & \frac{1}{4} & 0 & 0 & 500 \\
0 & 1 & -\frac{1}{10} & \frac{2}{5} & 0 & 400 \\
0 & -\frac{7}{2} & \frac{5}{4} & 0 & 1 & 2500
\end{array}
\]

\(-\frac{1}{2} R_2 + R_1 \rightarrow R_1\)

\(\frac{7}{2} R_2 + R_3 \rightarrow R_3\)

\[
\begin{array}{cccccc}
1 & 0 & \frac{3}{10} & -\frac{1}{5} & 0 & 300 \\
0 & 1 & -\frac{1}{10} & \frac{2}{5} & 0 & 400 \\
0 & 0 & \frac{9}{10} & \frac{7}{5} & 1 & 3900
\end{array}
\]

\(\text{x}_1, \text{x}_2 \text{ basic}\)

So, as before, \(P = 3900\) is maximum at \(\text{x}_1 = 300,\)

\(\text{x}_2 = 400, \ S_1 = 0, \ S_2 = 0.$$
Ex. 2. Maximise

\[ P = 16x_1 + 5x_2 + 20x_3 \]

Subject To

\[ 8x_1 + x_2 + 2x_3 \leq 1000 \]
\[ 2x_1 + x_2 + 7x_3 \leq 2000 \]

and \[ x_1, x_2, x_3 \geq 0. \]

Solution: We introduce slack variables so that our problem constraints become

\[ 8x_1 + x_2 + 2x_3 + s_1 = 1000 \]
\[ 2x_1 + x_2 + 7x_3 + s_2 = 2000 \]

with \[ s_1, s_2 \geq 0 \]

We rewrite the system as a simplex tableaux.

So our initial tableaux is
\[
\begin{bmatrix}
8 & 1 & 2 & 1 & 0 & 0 & 125 \\
0 & 3/4 & 13\frac{1}{2} & -\frac{1}{4} & 1 & 0 & 1750 \\
0 & -3 & -16 & 2 & 0 & 1 & 2000 \\
\end{bmatrix}
\]

Choose this column as pivot column.

\[
\begin{bmatrix}
1 & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & 0 & 0 & 125 \\
2 & 1 & 7 & 0 & 1 & 0 & 2000 \\
-16 & -5 & -20 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

-2R₁ + R₂ → R₂

16R₁ + R₃ → R₃

\[
\begin{bmatrix}
1 & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} & 0 & 0 & 125 \\
0 & \frac{3}{4} & \frac{13}{2} & -\frac{1}{4} & 1 & 0 & 1750 \\
0 & -3 & -16 & 2 & 0 & 1 & 2000 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
8 & 1 & 2 & 1 & 0 & 0 & 125 \\
0 & 3/4 & 13\frac{1}{2} & -\frac{1}{4} & 1 & 0 & 1750 \\
0 & -3 & -16 & 2 & 0 & 1 & 2000 \\
\end{bmatrix}
\]

\[\frac{125}{1/8} = 1000\]

\[\frac{1750}{3/4} = \frac{7000}{3} > 1000\]

8R₁ → R₁.
\[
\begin{bmatrix}
8 & 1 & 2 & 1 & 0 & 0 & 1000 \\
-6 & 0 & 5 & -1 & 1 & 0 & 1000 \\
24 & 0 & -10 & 5 & 0 & 1 & 5000 \\
\end{bmatrix}
\]
\[
\frac{1000}{2} = 500
\]
\[
\frac{1000}{5} = 200 \\
\text{since } 200 < 500
\]
\[
\frac{1}{5} R_2 \rightarrow R_2
\]
\[
\begin{bmatrix}
8 & 1 & 2 & 1 & 0 & 0 & 1000 \\
-6/5 & 0 & 1 & -1/5 & 1/5 & 0 & 200 \\
24 & 0 & -10 & 5 & 0 & 1 & 5000 \\
\end{bmatrix}
\]
\[
10 R_2 + R_3 \rightarrow R_3
\]
\[
\begin{bmatrix}
52/5 & 1 & 0 & 7/5 & 2/5 & 0 & 600 \\
-6/5 & 0 & 1 & -1/5 & 1/5 & 0 & 200 \\
12 & 0 & 0 & 3 & 2 & 1 & 7000 \\
\end{bmatrix}
\]

Hence \( 12 x_1 + 3 s_1 + 2 s_2 + P = 7000 \)

Taking \( x_1 = s_1 = s_2 = 0 \) as they are non-basic variables.

Our solution is \( P = 7000 \) is the optimal value achieved at \( x_1 = 0, x_2 = 600, x_3 = 200 \)

(With \( s_1 = 0, s_2 = 0 \))