Some solutions to Problem Set 2.

1. Here is one interpretation of the metrics: \( d_M \) gives the shortest distance between two points if you can only travel vertically and along the \( x_1 \)-axis; \( d_K \) gives the shortest distance if you can only travel along straight lines through the origin.

(b) The sequence \((x_n)\) does not converge in \( (\mathbb{R}^2, d_M) \) since if \( x = (a, b) \in \mathbb{R}^2 \), then
\[
d_M(x_n, x) = \left\| \frac{n}{n+1} - a \right\| + \frac{n}{n+1} |b| \geq \frac{1}{2} \text{ for all } n \geq 1 \text{ with } \frac{n}{n+1} \neq a.
\]
However, \((x_n)\) converges to \( x = (1, 1) \) with respect to the metric \( d_K \) since
\[
d_K(x_n, x) = \left\| \left( \frac{n}{n+1} - 1, \frac{n}{n+1} - 1 \right) \right\| = \sqrt{\frac{2}{n+1}} \to 0 \text{ as } n \to \infty.
\]
(c) The sequence \((x_n)\) converges to \( x = (0, 0) \) in \((\mathbb{R}^2, d_M)\) since \( d_M(x_n, x) = \left| \frac{1}{n} - 0 \right| + \left( \sqrt{n+1} - \sqrt{n} \right) + |0| = \frac{1}{n} + \frac{1}{\sqrt{n+1} + \sqrt{n}} \to 0 \) as \( n \to \infty \). It also converges to \( x = (0, 0) \) in \((\mathbb{R}^2, d_K)\) since
\[
d_K(x_n, x) = \left\| \left( \frac{1}{n}, \sqrt{n+1} - \sqrt{n} \right) \right\| = \sqrt{\frac{1}{n^2} + \frac{1}{(\sqrt{n} + 1 + \sqrt{n})^2}} \leq \sqrt{\frac{2}{n}} \to 0.
\]

2. Using the triangle inequality twice we have \( d(x_n, y_m) \leq d(x_n, x) + d(x, y) + d(y, y_m) \), hence \( d(x_n, y_m) - d(x, y) \leq d(x_n, x) + d(y, y_m) \). Similarly, we obtain \( d(x_n, y_m) - d(x, y) \leq d(x_n, x) + d(y, y_m) \), hence \( |d(x_n, y_m) - d(x, y)| \leq |d(x_n, x) + d(y, y_m)| \). As \( n \to \infty \), we have \( d(x_n, x) \to 0 \) and \( d(y_m, y) \to 0 \), hence \( |d(x_n, y_m) - d(x, y)| \to 0 \), i.e. \( d(x_n, y_m) \to d(x, y) \).

3. (a) Consider \( \mathbb{R} \) with the metrics \( d(x, y) = |x - y| \) and \( \bar{d}(x, y) = \frac{|x - y|}{1 + |x - y|} \). Then \( d \) and \( \bar{d} \) are equivalent. However, they are not Lipschitz equivalent since if \( x_n = n \) and \( x = 0 \), then \( d(x_n, x) = n \) and \( \bar{d}(x_n, x) = \frac{n}{n+1} < 1 \) and so there is no constant \( C \) such that \( d(x_n, x) \leq C \cdot \bar{d}(x_n, x) \) for all \( n \).

(b) Define \( d_\infty(x, y) = \max\{ |x_1 - y_1|, \ldots, |x_n - y_n| \} \) for \( x, y \in \mathbb{R}^n \). Note that for any \( r \geq 1 \),
\[
d_\infty(x, y) \leq d_r(x, y) \leq n^{1/r} d_\infty(x, y).
\]

Hence if \( q > p \geq 1 \), then
\[
d_q(x, y) \leq n^{1/q} d_\infty(x, y) \leq n^{1/q} d_p(x, y)
\]
and
\[
d_p(x, y) \leq n^{1/p} d_\infty(x, y) \leq n^{1/p} d_q(x, y).
\]

Combining the above inequalities one gets
\[
n^{-1/q} d_q(x, y) \leq d_p(x, y) \leq n^{1/p} d_q(x, y).
\]

4.

(a) \( A \) is open in \( X \) but not in \( \mathbb{R} \), \( A \) is not closed in both spaces

(b) \( B \) is open in \( X \), not open in \( \mathbb{R} \), not closed in \( X \) and not closed in \( \mathbb{R} \)

(c) \( C \) is not open and not closed in \( X \). It is not open and not closed in \( \mathbb{R} \).

(d) \( D \) is not open in both spaces, and closed in both spaces.

(e) \( E \) open in both spaces, not closed in both spaces.

5. \( A \) is neither open nor closed, \( B \) is closed, \( C \) is neither open nor closed, \( D \) is neither open nor closed, \( E \) neither open nor closed.

6.

(a) \( A^o = A, \quad \overline{A} = \{(x, y) \mid x \geq 0\}, \quad \partial A = \{(x, y) \mid x = 0, y \in \mathbb{R}\} \cup \{(x, y) \mid x \geq 0, y = 0\}\)

(b) \( B^o = \emptyset, \quad B = \overline{B}, \quad \partial B = B \)
(c) \( C^0 = A, \overline{C} = \{ (x,y) \in \mathbb{R}^2 \mid x \geq 0 \}, \partial C = \{ (x,y) \mid x = 0, y \in \mathbb{R} \} \cup \{ (x,y) \in \mathbb{R}^2 \mid x \geq 0, y = 0 \} \)

(d) \( D^0 = \emptyset, \overline{D} = \mathbb{R}^2, \partial D = \mathbb{R}^2 \).

(e) \( F^0 = \{ (x,y) \mid x \neq 0 \text{ and } y < 1/x \}, \overline{F} = F \cup \{ (x,y) \mid x = 0, y \in \mathbb{R} \}, \partial F = \{ (x,y) \mid x \neq 0 \text{ and } y = 1/x \} \cup \{ (x,y) \mid x = 0, y \in \mathbb{R} \}. \)

7. The answer to both questions is No! For example consider \( A = \mathbb{Q} \) in \( \mathbb{R} \) equipped with the usual metric. Then \( A^0 = \emptyset \) but \( \overline{A} = \mathbb{R} \) so that \((\overline{A})^0 = \mathbb{R}^0 = \mathbb{R} \). Also \( A^0 = \emptyset \).

8. \( \bigcup_{i \in I} A_i^0 \subseteq \left( \bigcup_{i \in I} A_i \right)^0 \): Let \( x \in \bigcup_{i \in I} A_i^0 \). Then \( x \in A_i^0 \) for some \( i \). Hence \( B(x,r) \subseteq A_i \) for some \( r > 0 \), and so \( B(x,r) \subseteq \bigcup_{i \in I} A_i \). Consequently, \( x \) is an interior point of \( \bigcup_{i \in I} A_i \).

- \( \bigcap_{i \in I} A_i \subseteq \bigcap_{i \in I} \overline{A}_i \): Let \( x \in \bigcap_{i \in I} A_i \). So for every \( B(x,r) \cap \bigcap_{i \in I} A_i \neq \emptyset \) which means that \( B(x,r) \cap A_i \neq \emptyset \) for all \( i \in I \). Hence \( x \) is an adherent point of \( A_i \) for all \( i \in I \), that is, \( x \in \overline{A}_i \) for all \( i \in I \). Therefore, \( x \in \bigcap_{i \in I} \overline{A}_i \).

- \( \left( \bigcap_{i \in I} A_i^{0}\right) \subseteq \bigcap_{i \in I} A_i \): Let \( x \in \left( \bigcap_{i \in I} A_i^{0}\right) \). So \( B(x,r) \subseteq \bigcap_{i \in I} A_i \). Hence \( B(x,r) \subseteq A_i \) for all \( i \in I \), that is \( x \in A_i \) for all \( i \in I \). Therefore, \( x \in \bigcap_{i \in I} A_i \).

- \( \bigcup_{i \in I} A_i \subseteq \bigcup_{i \in I} \overline{A}_i \): Let \( x \in \bigcup_{i \in I} A_i \). Then \( x \in A_j \) for some \( j \in I \), and so \( B(x,r) \cap A_j \neq \emptyset \). Consequently, \( B(x,r) \cap \left( \bigcup_{i \in I} A_i \right) \neq \emptyset \). This means that \( x \) is an adherent point of \( \bigcup_{i \in I} A_i \), i.e., \( x \in \bigcup_{i \in I} A_i \).

9. (a) Since \( A \subseteq \overline{A} \) and \( \partial A \subseteq \overline{A} \), \( A \cup \partial A \subseteq \overline{A} \). Conversely, if \( x \in \overline{A} \) and \( x \notin A \) then \( x \in \overline{X \setminus A} \) so that \( x \in \partial A \). Consequently, \( \overline{A} \subseteq A \cup \partial A \).

(b) If \( x \in \partial A \), then \( x \in \overline{A} \) and \( x \in \overline{X \setminus A} \). Since \( x \in \overline{X \setminus A} \), for every \( r > 0 \), \( B(x,r) \cap (X \setminus A) \neq \emptyset \), implying that \( x \notin A^0 \). Hence \( x \in \overline{A} \setminus A^0 \). Conversely, if \( x \in \overline{A} \setminus A^0 \), then for every \( r > 0 \), \( B(x,r) \cap [X \setminus A] \neq \emptyset \) since otherwise \( B(x,r) \subseteq A \) for some \( r \) and then \( x \in A^0 \). Hence \( x \in \overline{X \setminus A} \) and \( x \in \partial A \).

(c) If \( \overline{A} = A \), then in view of (b), \( \partial A = \overline{A} \setminus A^0 = A \setminus A^0 \). If \( \partial A = A \setminus A^0 \), then \( \partial A \subseteq A \) and in view of (a), \( \overline{A} = A \cup \partial A \subseteq A \) and so \( A \) is closed.

(d) If \( A \) is open, then using (b), \( \partial A = \overline{A} \setminus A^0 = \overline{X \setminus A} \). Conversely, if \( \partial A = \overline{X \setminus A} \), then in view of the second part of (b), \( A^0 = A \setminus \partial A = A \setminus [\overline{A} \setminus A] = A \setminus [\overline{A} \cap (X \setminus A)] = (A \setminus \overline{A}) \cup [A \setminus (X \setminus A)] = A \).

10. (a) Let \( a \in A \). We have to show that \( a \) is an interior point of \( A \). Since \( B \) is non-empty, there is \( b \in B \). So \( (a,b) \in A \times B \). Since \( A \times B \) is open, \( (a,b) \) is an interior point of \( A \times B \) and there exists \( r > 0 \) such that \( B((a,b),r) \subseteq A \times B \). For any \( x \in B(a,r) \), we have

\[
d((a,b),(x,b)) = d_X(a,x) + d_Y(b,b) = d_X(a,x) < r,
\]

so that \((x,b) \in B((a,b),r) \subseteq A \times B \). Hence \( B(a,r) \subseteq A \) and \( a \) is an interior point of \( A \). Consequently, any point in \( A \) is an interior point of \( A \) which means that \( A \) is open.

(b) Let \( x \in X \) be an adherent point of \( A \). We have to show that \( x \in A \). There exists a sequence \( \{x_n\} \) such that \( x_n \in A \) and \( x_n \to x \) in \( X \). Since \( B \) is non-empty, there is \( y \in B \). For the sequence \( \{x_n,y\} \in A \times B \), we have

\[
d((x_n,y),(x,y)) = d_X(x_n,x) + d_Y(y,y) = d_X(x_n,x) \to 0,
\]

showing that \((x,y) \) is an adherent point of \( A \times B \). Since \( A \times B \) is closed in \( X \times Y \), \((x,y) \in A \times B \), that is, \( x \in A \), as required.