**Theorem.** The structure of $K_c$ is governed by the orbit of $z = 0$:

- If $Q^n_c(0) \not\to \infty$ then $K_c$ is connected.
- If $Q^n_c(0) \to \infty$ then $K_c$ is a Cantor set and is completely disconnected.

In this case $K_c = J_c$ and the dynamics of $Q_c$ on this set is conjugate to the shift map on two symbols.

- This theorem suggests another set worth plotting:

**Definition.** The Mandelbrot set $\mathcal{M}$ is the set:

$$\mathcal{M} = \{ c \in \mathbb{C} \mid K_c \text{ is connected} \}$$

By the above, this is equivalent to:

$$\mathcal{M} = \{ c \in \mathbb{C} \mid |Q^n_c(0)| \not\to \infty \}$$

**Note:**

- While $K_c$ is a set of $z$-values — it lives in state-space.
- $M$ is a set of $c$-values — it lives in parameter space.

- We can use the above, together with the escape criterion to build an algorithm to find $\mathcal{M}$
• Reminder...

**Corollary (Escape criterion).** Suppose \( \exists k \geq 0 \) such that 
\[ |Q^k_c(z)| > \max\{|c|, 2\} \] then \( |Q^n_c(z)| \to \infty \).

• Since \( Q_c(0) = c \) it follows that:

**Corollary. To which points are not in \( M \):**

• If \( |c| > 2 \) or \( |Q^k(c)| > 2 \) (for some \( k > 0 \)) then \( |Q^n_c(0)| \to \infty \)

• Hence \( c \notin M \)

**Algorithm for the Mandelbrot set:**

• Pick a grid of points in \( \mathbb{C} \) around \( z = 0 \) and some maximum number of iterations, \( N \).

• For each point \( c \) in the grid, compute the first \( N \) points of the orbit of \( z = 0 \) under \( Q_c \).

• If \( Q^k_c(0) \) “escapes” for some \( k \leq N \) then that \( c \) value is not in \( M \).

• Otherwise the \( c \) value is probably in \( M \).

• Colour points in \( M \) black, and other points white.

• Alternatively we can make prettier pictures by colouring those points that escape according to how fast they escape.
• Like $K_c$, the Mandelbrot set is incredibly complex.
• It is self-similar and is a fractal.
So what does this image mean?

**Definition.** An alternative definition of $\mathcal{M}$ is:

$$\mathcal{M} = \{c \in \mathbb{C} \mid Q_c(z) \text{ has an attracting fp or pp}\}$$

- Say $Q_c(z)$ has an attracting fixed or periodic point.
- Then around this point there is a neighbourhood of points whose orbits converge to this point.
- All these points lie in $K_c$.
- Hence $K_c$ is not a Cantor set, and so (by the above theorem) must be simply connected.
- So $c$ is in $\mathcal{M}$.

- Indeed each “blob” or “bulb” of $\mathcal{M}$ corresponds to a region of $c$ in which $Q_c$ has attracting periodic points of a given period.
• While this picture doesn’t look like it, $M$ is actually connected.
• We can compute some of these bulbs exactly.

**Proposition.** *Period 1 and 2 bulbs:*

• $Q_c(z)$ has an attracting fixed point inside the region defined by

$$c = \frac{1}{2}e^{i\theta} - \frac{1}{4}e^{2i\theta}$$

• $Q_c(z)$ has an attracting 2-cycle point inside the region defined by

$$c = \frac{1}{4}e^{i\theta} - 1$$

• Other bulbs we have to check numerically.

• Though there is a general theorem for which bulbs are which period (we aren’t going to do it).
Proof: (fixed points)

- Denote the fixed points of $Q_c$ by $p_{\pm}$.
- We require $|Q'_c(p_{\pm})| = 2|p_{\pm}| < 1$.
- So we have an attracting fp if $p_{\pm} = \rho e^{i\theta}$ with $\rho < 1/2$.
- Substituting this into $Q(z) = z$ gives
  \[
  \rho^2 e^{2i\theta} + c = \rho e^{i\theta}
  \]
  or
  \[
  c = \rho e^{i\theta} - \rho^2 e^{2i\theta}
  \]
- Which corresponds to $c$ inside the region defined by
  \[
  c = \frac{1}{2} e^{i\theta} - \frac{1}{4} e^{2i\theta}
  \]

Proof: (2-cycle)

- The 2-cycle of $Q_c$ is given by $q_{\pm} = -\frac{1}{2} \pm \frac{1}{2} \sqrt{-3 - 4c}$.
- We have $|(Q^2_c)'(z_0)| = |Q'_c(z_0)||Q'_c(z_1)|$.
- So for the 2-cycle we need $4|q_+||q_-| = 4|q_+ q_-| < 1$
- Now $q_+ q_- = c + 1$.
- Hence we need $|c + 1| < 1/4$, which corresponds to $c$ inside the region
  \[
  c = \frac{1}{4} e^{i\theta} - 1
  \]
• The bifurcation diagrams we studied earlier correspond to the slice of $\mathcal{M}$ along the real axis.

• The period 3 bulb in $\mathcal{M}$ corresponds to the period 3 window in the bifurcation diagram
- Here is the period 3 bulb and corresponding window of the bifurcation diagram:
• We can also generate Mandelbrot sets for other functions:

• These are the Mandelbrot sets of $z^3 + c$, $z^4 + c$ and $z^5 + c$. 
• We can find the period-1 bulbs of the Mandelbrot sets of $F(z) = z^n + c$:

**Theorem.** The period-1 bulb of the Mandelbrot set of $z^n + c$ is given by the curve:

$$c = n^{-1/(n-1)}e^{i\theta} - n^{-n/(n-1)}e^{ni\theta}$$

• We do this in the same way we did for $z^2 + c$.
• For a fixed point, we require that $|F'(z_0)| < 1$
• This gives the equation

$$|F'(z_0)| = n|z_0|^{n-1} < 1 \quad \text{or} \quad |z_0| < n^{-1/(n-1)}.$$  

• Hence if we have a fixed point of the form $z_0 = \rho e^{i\theta}$ with $\rho < n^{-1/(n-1)}$, then it is attracting.
• Substituting this into $z^n + c = z$ to obtain $c$ we get:

$$c = \rho e^{i\theta} - \rho^n e^{ni\theta}$$

• As so the required $c$ values lie inside the curve:

$$c = n^{-1/(n-1)}e^{i\theta} - n^{-n/(n-1)}e^{ni\theta}$$